A topological model for the HOMFLYPT polynomial (arXiv:2405.03679)

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The HOMFLYPT polynomial $P_L(a, z)$ for $L \subset S^3$ is defined by the skein relation:

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One way to approach is to construct a **topological model**: Recover $P_L(a, z)$ as an intersection pairing of homology classes of curves on a covering of a configuration space.

Theorem (Anghel-L. 2024)

Let $\Theta_H(D) \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ be the state sum of graded intersections between explicit Lagrangian submanifolds in a fixed configuration space on a Heegaard surface Σ for L:

$$\Theta_H(D)(a,z) := \sum_{\substack{\sigma_H^K \text{ a renormalized} \\ \text{Kauffman state} \\ \text{associated to a} \\ \text{Jaeger state } \sigma_H}} sgn(\sigma_H^K) \cdot a^{i^a(\sigma_H^K)} \cdot z^{i^z(\sigma_H^K)}.$$

$$\cdot \ll \mathscr{F}^H(\sigma_H^K), \mathscr{L}^H(\sigma_H^K) \gg_{\alpha_H^{\sigma_H^K}}$$

Then this topological model recovers the HOMFLYPT invariant:

$$\Theta_H(D) = P_L(a,z).$$

Known topological models

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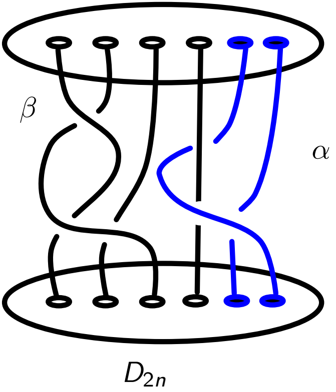
- Anghel has defined topological models for the colored Jones polynomial [Anghel, 2022], the colored Alexander polynomials [Anghel, rier], and the Witten-Reshetikhin-Turaev invariants [Anghel, 2023].

The Lawrence representation [Lawrence, 1990]

Let C_n be the unordered configuration space of n particles on D_{2n}.



$$\phi : \pi_1(C) \to \mathbb{Z}\langle q, t\rangle$$
$$\alpha \mapsto q^a t^b$$

$$\begin{array}{c} B_{2n} \curvearrowright \widetilde{C_n} \\ \big\downarrow \\ C_n \end{array}$$

Bigelow's noodles and forks

- Intersection pairing

$$\langle \widetilde{A}, \widetilde{B} \rangle = \sum_{a,b \in \mathbb{Z}} (q^a t^b \widetilde{A}, \widetilde{B}) q^a t^b$$

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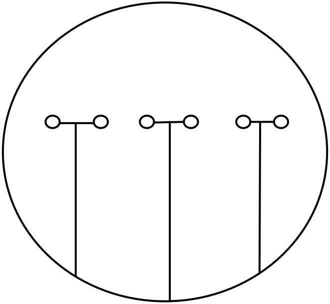
- A homological model for the Jones polynomial [Bigelow, 2002]

$$V_\beta'(q) := \lambda \langle \widetilde{S}, \beta \widetilde{T} \rangle \Big|_{t=-q^{-1}}$$

recovers the Jones polynomial $J_L(q)$.



standard fork

Question

What about the HOMFLYPT polynomial?

Recall the skein relation:

$$aP(\nwarrow\!\!\!\!\!\nearrow) - a^{-1}P(\nwarrow\!\!\!\!\!\nearrow) = zP(\upharpoonleft\,\upharpoonright) \qquad\qquad P(U) = 1 \tag{2}$$

- The representation theory is not well understood.
- We would like to reduce the dependence on a braid representation [Abel, 2017], [Khovanov and Rozansky, 2008].

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Idea: Work with a state sum from a diagram D of L.

- Use of a Heegaard surface (with punctures) as the base surface of the configuration space.
- Dependence on D rather than $\widehat{\beta}$.

Outline of the construction

Jaeger's state sum definition of $P_L(a,z)$ writes it as

$$P_L(a,z) = \sum_{\substack{\sigma_{H,P} \text{ a Jaeger state} \\ \text{on the diagram } D}} \sigma_{H,P}(a,z) \left(\frac{a - a^{-1}}{z} \right)^{|\sigma_{H,P}|-1}$$

- We interpret $\sigma_{H,P} = \mathrm{sgn}(\sigma_H^K) \cdot a^{i^a(\sigma_H^K)} \cdot z^{i^z(\sigma_H^K)}$ as geometric intersections of curves on Σ.

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- We interpret $\sigma_{H,P} = \mathrm{sgn}(\sigma_H^K) \cdot a^{i^a(\sigma_H^K)} \cdot z^{i^z(\sigma_H^K)}$ as geometric intersections of curves on Σ.
- What remains is to recover

$$\left(\frac{a - a^{-1}}{z} \right)^{|\sigma_{H,P}|-1}$$

as an intersection pairing of homology classes.

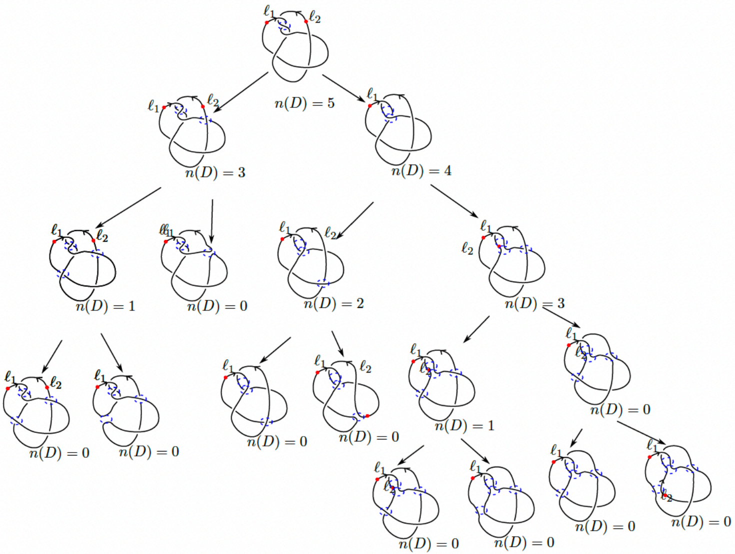


FIGURE 20. An example of the binary tree T_D.

Lemma

Given a link diagram D with ℓ components, there exists a Kauffman state σ on D with $|\sigma| = \ell$ state circles in S_σ.

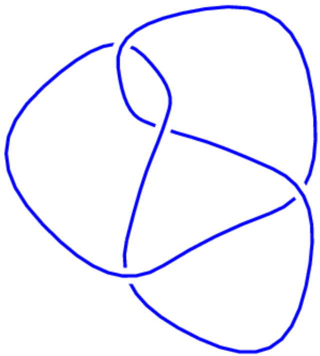
$$P_L(a,z) = \sum_{\substack{\sigma_{H,P} \text{ a Jaeger state} \\ \text{on the diagram } D}} \sigma_{H,P}(a,z) \left(\frac{a - a^{-1}}{z}\right)^{|\sigma_{H,P}|-1}$$

$$= \sum_{\substack{\sigma^K_{H,P} \text{ a Kauffman state} \\ \text{associated to the Jaeger state} \\ \text{on the diagram } D}} \sigma_{H,P}(a,z) \left(\frac{a - a^{-1}}{z}\right)^{|\sigma^K_{H,P}|-1}$$

The problem reduces to defining $\ll, \gg$ to get the evaluation of the Kauffman bracket on unlinks.

A local system on the (punctured) Heegaard surface Σ

A Heegaard surface Σ from D [Ozsváth and Szabó, 2003].



- Mark points x_ℓ, x_r, y_ℓ, y_r on the surface at every crossing χ
- Define additional punctures P at every crossing χ
- Remove an additional special puncture s

Let n be the number of crossings of D. Consider the configuration space of $2n$ particles on $\Sigma' = \Sigma \setminus (P \cup \{s\})$.

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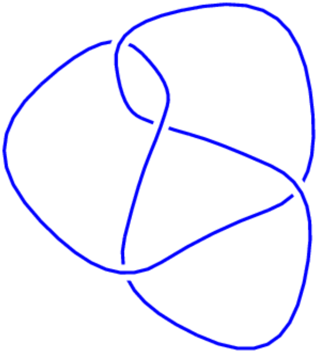

- Define a pair of basepoints at every crossing.
- Define loops based at b_j at every crossing.

Let $\gamma_j' := (b_1, b_2, \ldots, \gamma_j, \ldots, b_{2n})$.
$\{[\gamma_j]\}$ define linearly-independent classes in $\pi_1(Conf_{2n}(\Sigma'))$.

Let $\widetilde{C\Sigma'}$ be the covering of $C\Sigma'$ corresponding to the kernel of the map $\Phi = \nu \circ p \circ ab$.

$$\begin{aligned} \Phi : \pi_1(C_{2n}(\Sigma')) &\to \mathbb{Z}^{8n} \\ \langle\{\gamma'_j\}\rangle &\mapsto \langle\{x_j\}_{j\in\{1,\ldots,8n\}}\rangle \end{aligned}$$

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$$\Phi : \pi_1(C_{2n}(\Sigma')) \to \mathbb{Z}^{8n}$$
$$\langle \{\gamma_j'\} \rangle \mapsto \langle \{x_j\}_{j \in \{1,\ldots,8n\}} \rangle$$

This is equipped with the intersection pairing [Anghel and Palmer, 2020]:

$$\ll \ , \ \gg : \mathscr{H}^{lf}_{2n} \otimes \mathscr{H}_{2n} \to \mathbb{Z}[x_1^{\pm 1}, \ldots, x_{8n}^{\pm 1}].$$

$$\ll H_1, H_2 \gg = \sum_{x \in X_1 \cap X_2} \alpha_x \cdot \Phi(\ell_x) \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_{8n}^{\pm 1}]. \qquad (3)$$

Goal: Define $\mathscr{F}(\sigma), \mathscr{L}(\sigma)$ and specialization α_Q^σ in $\widetilde{C\Sigma'}$ so that

$$\ll \mathscr{F}(\sigma), \mathscr{L}(\sigma) \gg_{\alpha_Q^\sigma} \text{ recovers } (q + q^{-1})^{|\sigma|}$$

Define $\mathscr{F}(\sigma), \mathscr{L}(\sigma)$ in $\widetilde{C\Sigma'}$ so that

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We first construct $F(\sigma)$: blue arcs and $L(\sigma)$: green ovals on Σ'.

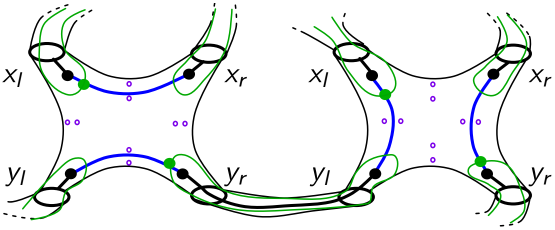

Figure: Construction of arcs and ovals from Kauffman state σ.

Example of $F(\sigma), L(\sigma)$

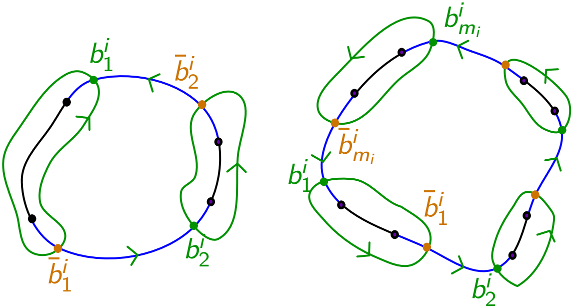

Figure: The collection of green ovals replacing the black arcs.

From $F(\sigma), L(\sigma)$ to $\mathscr{F}(\sigma), \mathscr{L}(\sigma)$

$$
\begin{tikzcd}
 & \{\mathscr{F}(\sigma), \mathscr{L}(\sigma)\} \subset \widetilde{C\Sigma'} \arrow[dd] \\
\{F(\sigma), L(\sigma)\} \subset \Sigma' \arrow[ur, "\text{lift}"] \arrow[dr, swap, "\substack{\text{encodes submanifolds} \\ \text{in } \widetilde{C\Sigma'} \\ \text{[Anghel and Palmer, 2020]}}"] & \\
 & C\Sigma'
\end{tikzcd}
$$

The Monodromy requirement

Recall the intersection pairing

$$\ll \ , \ \gg : \mathscr{H}_{2n}^{lf} \otimes \mathscr{H}_{2n} \to \mathbb{Z}^{8n}.$$

To ensure we do recover $(q+q^{-1})^{|\sigma|}$ when evaluating $\ll \mathscr{F}(\sigma), \mathscr{L}(\sigma) \gg$, we define a specialization of the intersection pairing so that the evaluation around each state circle is the same.

Lemma

There is a change of coefficients α_Q^σ

$$\alpha_Q^\sigma : \qquad \mathbb{Z}[x_{f(1)}^{\pm 1}, \ldots, x_{f(|\sigma|)}^{\pm 1}] \quad \to \quad \mathbb{Z}[Q^{\pm 1}]$$

which satisfies the Monodromy requirement.

Computing $\ll \mathscr{F}(\sigma), \mathscr{L}(\sigma) \gg_{\alpha_Q^\sigma}$

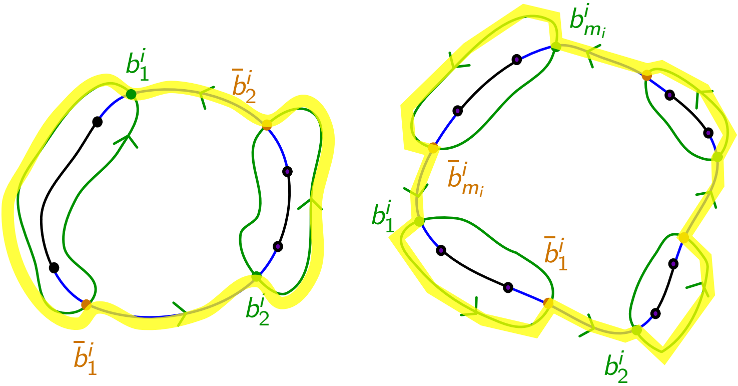

Figure: A loop ℓ_x.

Non semi-simple adjustment for the HOMFLYPT

We similarly construct $\mathscr{F}(\sigma_H^K), \mathscr{L}(\sigma_H^K)$ for evaluation in the intersection pairing $\ll \, , \, \gg_{\alpha_H^{\sigma_H^K}}$, with a slight modification to count one fewer circle.

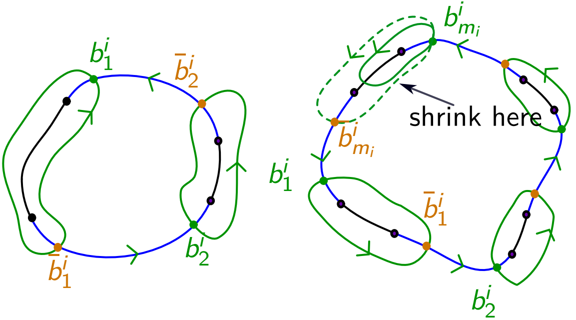

Figure: Homology classes for HOMFLYPT polynomial

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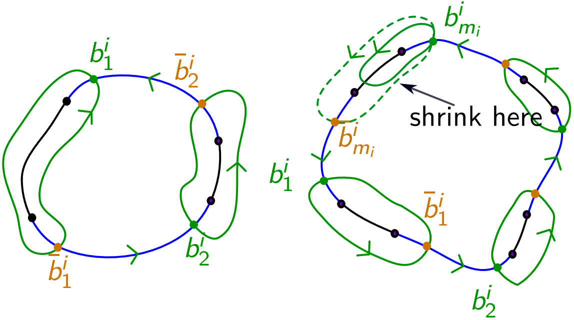

Figure: Homology classes for HOMFLYPT polynomial

- We use a change-of-coefficients $\alpha_H^{\sigma_H^K}$ distinct from α_J^{σ}.

Work in Progress

- Study the relationship to knot Floer homology: We can define an injective map from Jaeger states to the Alexander Kauffman states used by Osvath-Szabo [Ozsváth and Szabó, 2003].
- Finding applications.

Aganagic's proposed construction of Floer theory

Aganagic proposes a unifying homological theory that unifies HF and Khovanov homology.

- Floer complex category:
  - Objects: intersection points of special branes $\leftrightarrow$ projective modules/resolutions
  - Morphisms: exact supersymmetric ground states $\leftrightarrow$ maps between complexes

[Aganagic-LePage-Rapcak, 2025] claims the theory recovers Floer theory and all $\mathfrak{sl}(m)$ homology theories.

Thank you for listening!

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