Witten's asymptotic expansion conjecture and quantum modularity for Gukov-Pei-Putrov-Vafa invariants

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Abstract

Problems:

1. Topology: derive asymptotic expansions of WRT invariants for 3-manifolds.
2. Number theory: establish quantum modularity for false theta functions.

Previous results: Cases which admit single integrals.

Main results:

1. for negative definite plumbed manifolds.
2. for general false theta functions.

New techniques: (admit multiple integrals)

- Poisson summation formula with sign.
- A framework of "modular series."

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§ 1 Main result for topology

§ 2 Proof strategy

§ 3 Main results for number theory

§ 4 Poisson summation formula with sign

§ 5 "Modular series"

§ 1 Main results for topology

Quantum invariants

- K : knot $\mapsto (J_N(K;q))_{N=1}^{\infty}$: colored Jones polynomial

Today!

- M : 3-mfd $\mapsto (Z_k(M))_{k=1}^{\infty}$: Witten-Reshetikhin-Turaev invariant

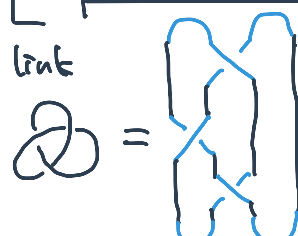
$\in \mathbb{C}$

▷ Physics : $\boxed{Z_k(M)} := \int e^{2\pi i (k-2) CS(A)} \mathcal{D}A$: path integral on $\{SU(2) \text{ connection on } M\} / \text{gauge equivalence}$

$$CS(A) := \frac{1}{8\pi^2} \int_M \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \text{ : Chern–Simons functional}$$

▷ Math : $M \xleftarrow{\text{Dehn surgery}} L \longmapsto F(L) \rightsquigarrow \boxed{Z_k(M)}$

L: link



Label $\cap\ \cup\ \times\ \times$ representation of the quantum group $U_q(\mathfrak{sl}_2)$, $q := e^{2\pi i/k}$

$F(L)$: link invariant

$Z_k(M) := \sum_{\text{labeling of } L} F(L)$

Asymptotics of quantum invariants

Volume conjecture (Kashaev, Murakami–Murakami)

K: hyperbolic knot $\Rightarrow$ $\left| J_N(K; e^{2\pi i/N}) \right| \underset{N\to\infty}{\sim} \exp\left(\frac{N}{2\pi} \mathrm{Vol}(S^3 \setminus K) \right)$

Witten's asymptotic expansion conjecture

Today!

$$Z_k(M) \underset{k\to\infty}{\sim} \sum_{\theta \in \mathbb{C}/\mathbb{Z}} e^{2\pi i k \theta} \; {}^{\exists} Z_\theta(k)$$

θ: Chern–Simons invariant of M

$Z_\theta(k) \in \bigcup_{p \geq 1} \mathbb{C}((k^{-\frac{1}{p}}))$

Proved for Seifert homology spheres:

Lawrence–Rozansky (1999) / Hikami (2006) / Andersen–Mistegård (2022) / Matsusaka–Terashima (2024)

Main thm 1 M : negative definite plumbed manifold

$$\Rightarrow \; Z_k(M) \underset{k\to\infty}{\sim} \sum_{\theta \in {}^{\exists} S \subset \mathbb{Q}/\mathbb{Z}} e^{2\pi i k \theta}\, {}^{\exists} Z_\theta(k)$$

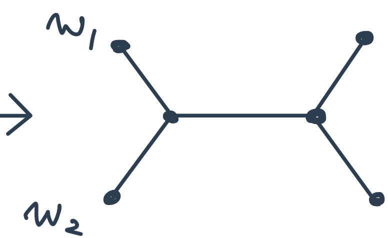
S: fin, explicit

$Z_\theta(k) \in \mathbb{C}((k^{-\frac{1}{2}}))$

- M : plumbed

a weighted tree

:⟺ M $\xleftarrow{\text{Dehn surgery}}$ 

⟷ 

- M : negative definite :⟺ the linking matrix is neg. def.

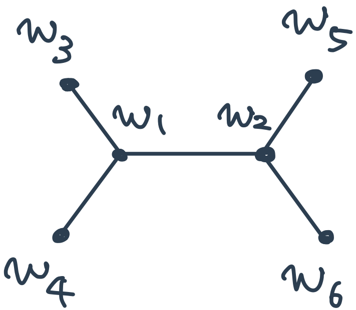
non-hyperbolic, $H_1(M, \mathbb{Q}) = 0$ $\supset$ negative definite $\supset$ Seifert homology spheres, lens spaces

Explicit form of $\mathcal{S}$

$$\mathcal{S} = \left\{ lk(\alpha,\alpha) - \frac{1}{4}\sum_{1\le s'<s} {}^t n_{s'} W^*_{s'} n_{s'} \bmod \mathbb{Z} \;;\; \begin{array}{l} \alpha \in H^1(M,\mathbb{Z}),\ V_1 \sqcup \cdots \sqcup V_s = V^\Downarrow, \\ n_{s'} \in \mathbb{Z}^{V_{s'}} \text{ satisfying some congruences} \end{array} \right\}$$

where
- $V^\Downarrow := \{$ vertices with $\deg \ge 3 \}$.
- W : the linking matrix.
- $(W^\Downarrow)^{-1} \in \mathrm{Sym}(\mathbb{Q}^{V^\Downarrow})$: $V^\Downarrow \times V^\Downarrow$-submatrix of W^{-1}.
- $W^*_{s'} \in \mathrm{Sym}(\mathbb{Q}^{V_{s'}})$: defined by using $W^\Downarrow$.

Ex H-graph & $H_1(M,\mathbb{Z})=0$ case.



$$\rightsquigarrow \quad \mathcal{S} = \left\{ 0,\ \frac{1}{4}\,{}^t n\, W^\vee n,\ \frac{n_2^2}{4 w_1^\vee P_1 P_2},\ \frac{n_1^2}{4 w_2^\vee P_1 P_2} \ \middle|\ n = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \in \mathcal{P}_1 \times \mathcal{P}_2 \right\}$$

where $w_1^\vee := w_1 - \frac{1}{w_3} - \frac{1}{w_4}$, $w_2^\vee := w_2 - \frac{1}{w_5} - \frac{1}{w_6}$,

$$W^\vee = \begin{pmatrix} w_1^\vee & 1 \\ 1 & w_2^\vee \end{pmatrix}, \quad P_1 := w_3 w_4, \quad P_2 := w_5 w_6,$$

$$\mathcal{P}_1 := \mathbb{Z} \setminus (w_3\mathbb{Z} \cup w_4\mathbb{Z}), \quad \mathcal{P}_2 := \mathbb{Z} \setminus (w_5\mathbb{Z} \cup w_6\mathbb{Z})$$

Comparison with asymptotic expansion conjecture

Q (joint work with Terashima)

$$\mathcal{S} \overset{?}{=} \{0\} \cup \{\text{Chern-Simons invariants of } M\}$$

Rem True for Seifert homology spheres.

§ 2 Proof strategy

Why difficult?

WRT inv

$$Z_k(M) \doteq \sum_{\mu \in (\mathbb{Z}-k\mathbb{Z})^r / 2k\mathbb{Z}^r} \zeta_{4k}^{Q(\mu)} F(\zeta_{2k}^{\mu_1}, \ldots, \zeta_{2k}^{\mu_r}) \quad \text{(in our case)}$$

⚠ Many k's appear ⚠

where

- $\zeta_k := e^{2\pi i/k}$,
- $Q : \mathbb{Z}^r \to \mathbb{Z}$: a quadratic form,
- $F(z_1, \ldots, z_r) \in \mathbb{Q}(z_1, \ldots, z_r)$

⇝ This expression does NOT yield the asymptotic expansion.

⇝ **Need** : Good expression of $Z_k(M)$.

Proof Strategy

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\overset{\text{WRT inv}}{Z_k} \arrow[d, dashed, "k\to\infty"', "?"] & & \overset{\text{GPPV inv}}{\hat{Z}(\tau)} \arrow[ll, "\tau \to \frac{1}{k}", "1"'] \arrow[d, mapsto, "2"', "\tau \mapsto -\frac{1}{\tau}"] \\
\sum_{\theta \in S} e^{2\pi i k \theta} Z_\theta(k) & & \hat{Z}^*(\tau) + \int \hat{Z}^{**}(\tau; \xi)\, d\xi \arrow[ll, "3", "\tau \to -k,\ k \to \infty"']
\end{tikzcd}$$

1 : Radial limit conjecture, conj'd by Gukov–Pei–Putrov–Vafa (2020)
proved by YM (2024)

2 : Quantum modularity

3 : Asymptotic expansions

(2 and 3): This talk!

Def M : a negative definite plumbed manifold

$\rightsquigarrow$ Gukov-Pei-Putrov-Vafa invariant

$$\hat{Z}_{b}(q;M) := q^{\Delta} \sum_{\ell \in b + 2W(\mathbb{Z}^V)} \boxed{F_\ell}\, q^{-{}^t\ell W^{-1} \ell/4} \in q^{\Delta}\mathbb{Z}[[q]]$$

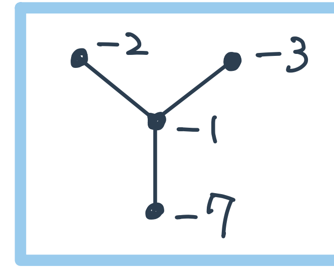
where

- $V := \{\text{vertices}\}$
- W : the adjacency matrix
- $b \in \mathrm{Spin}^c(M) \cong \dfrac{(\deg v)_v + 2\mathbb{Z}^V}{2W(\mathbb{Z}^V)}$
- $\Delta := -\dfrac{\operatorname{tr} W + |V|}{4}$
- $\boxed{F_\ell} := \displaystyle\prod_{v \in V} \mathrm{PV}\int_{|z_v|=1} (z_v - z_v^{-1})^{2-\deg v} \frac{z_v^{-\ell_v}\, dz_v}{2\pi i\, z_v}$

- Cauchy principal value

$$\mathrm{PV}\int_{|z|=1} := \lim_{\varepsilon \to 0} \frac{1}{2}\left(\int_{|z|=1-\varepsilon} + \int_{|z|=1+\varepsilon}\right)$$

Example : a Brieskorn homology sphere $\Sigma(2,3,7)$



- $W = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -2 & & \\ 1 & & -3 & \\ 1 & & & -7 \end{pmatrix}$, $\det W = 1$
- $H_1(M,\mathbb{Z}) \cong \mathbb{Z}^4/W(\mathbb{Z}^4) = 0$
- $\mathrm{Spin}^c(M) \cong (\delta + 2\mathbb{Z}^4)/2W(\mathbb{Z}^4) = (\delta + 2\mathbb{Z}^4)/2\mathbb{Z}^4$, $\delta := (3,1,1,1)$
- WRT inv $\quad Z_k(M) = \dfrac{-\zeta_{4k}^{q}}{2(2k)^2(\zeta_{2k}-\zeta_{2k}^{-1})} \displaystyle\sum_{\mu \in (\mathbb{Z}\setminus k\mathbb{Z})^4/2k\mathbb{Z}^4} \zeta_{4k}^{{}^t\mu W \mu} \dfrac{\prod_{2\le i\le 4}(\zeta_{2k}^{\mu_i}-\zeta_{2k}^{-\mu_i})}{\zeta_{2k}^{\mu_1}-\zeta_{2k}^{-\mu_1}}$

① $\uparrow q \to \zeta_k$

- GPPV inv $\quad \hat{Z}_\delta(q;M) = q^{\frac{5}{2}} \displaystyle\sum_{m=0}^{\infty} \chi(m)\, q^{\frac{m^2-1}{168}} = \frac{1}{2} q^{\frac{5}{2}} \sum_{m=-\infty}^{\infty} \mathrm{sgn}(m)\, \chi(m)\, q^{\frac{m^2-1}{168}}$

where $\chi : \mathbb{Z}/84\mathbb{Z} \to \{0, \pm 1\}$

$$1, -13, -29, 41 \mapsto 1$$
$$-1, 13, 29, -41 \mapsto -1$$
$$\text{otherwise} \mapsto 0$$

rank 1
false theta function !

rank $\in \mathbb{Z}_{>0}$ in general $\rightsquigarrow$ Difficult!

- CS inv $\mathcal{S} = \left\{ -\frac{n^2}{168} \bmod \mathbb{Z} \;\middle|\; n \in (\mathbb{Z}/42\mathbb{Z})^\times \right\}$

- ② Quantum modularity for GPPV inv

admits $\tau \to -k$ & $k \to \infty$

$$\hat{Z}_{\mathcal{S}}\left(-\frac{1}{\tau}\right) = \tilde{q}^{\frac{5}{2}-\frac{1}{168}} \cdot 16i\sqrt{\frac{84i}{\tau}} \sum_{\theta \in \mathcal{S}} Z^*_\theta(q; M) + \Omega(\tau)$$

where ▷ $q := e^{2\pi i \tau}$, $\tilde{q} := e^{2\pi i \frac{-1}{\tau}}$

rank 1 false theta

▷ $$Z^*_\theta(q; M) := \sum_{\substack{n \in \mathbb{Z}, \\ \theta \equiv -\frac{n^2}{168} \bmod \mathbb{Z}}} (-1)^n \sin\frac{\pi n}{2} \sin\frac{\pi n}{3} \sin\frac{\pi n}{7}\, q^{\frac{n^2}{168}}$$

▷ $$\Omega(\tau) := \int_0^{\infty + i\varepsilon} \tilde{q}^{84\xi^2} G(e^{2\pi i \xi})\, d\xi, \quad G(z) := \frac{(z^{\frac{1}{2}} - z^{-\frac{1}{2}})(z^{\frac{1}{3}} - z^{-\frac{1}{3}})(z^{\frac{1}{7}} - z^{-\frac{1}{7}})}{z - z^{-1}}$$

converges for $\tau = -k$

- ③ Asymptotic expansion $$Z_k(M) \underset{k\to\infty}{\sim} \sum_{\theta \in \mathcal{S}} e^{2\pi i k\theta} \underbrace{Z^*_\theta(k)}_{\in \mathbb{C}[\sqrt{k}]} + \underbrace{\tilde{\varphi}(k)}_{\in \mathbb{C}((\frac{1}{\sqrt{k}}))}$$

§ 3 Main results for number theory

Def A false theta function of rank r & depth $\leq d$ is

$$\widetilde{\Theta}(\tau) := \sum_{m \in \alpha + \mathbb{Z}^r} \operatorname{sgn}(Am)\, P(m)\, q^{Q(m)},$$

where $Q : \mathbb{Q}^r \to \mathbb{Q}$: pos def quad form, $A \in M_{s,r}(\mathbb{Z})$: rank d,
$\alpha \in \mathbb{Q}^r$, $P(x) \in \mathbb{Q}[x]$, $\operatorname{sgn}(y) := \operatorname{sgn}(y_1) \cdots \operatorname{sgn}(y_s)$, $q := e^{2\pi i \tau}$.

Main Thm 2 (② Quantum modularity for false theta functions)

$$\sqrt{\frac{i}{\tau}}^{\,r}\, \widetilde{\Theta}\left(-\frac{1}{\tau}\right) = \sum_{1 \leq i \leq M} {}^{\exists}\widetilde{\Theta}_i(\tau)\, {}^{\exists}\Omega_i(\tau) + {}^{\exists}\Omega(\tau)$$

hol func on $\widetilde{\mathbb{C} \setminus \{0\}}$

false theta func of rank $\leq r$ & depth $\leq d$

Previous works

- **Rank 1 case**
  - ▷ Lawrence-Zagier (1999): for radial limits
  - ▷ Creutzig-Milas (2014): first proof
  - ▷ Andersen-Mistegård (2022), Han-Li-Sauzin-Sun (2023), Creu-Fantini-Goswami-Osbern-Wheeler (2025): proof by resurgence theory
- **General rank case**
  - ▷ Bringmann-Nazaroglu (2019): modular completion for $P=1$
- **Formulation of quantum modularity**
  - ▷ Zagier (2010), Garoufalidis-Zagier (2023), (2024), Wheeler's thesis (2023)

Main Thm 3 (③ Asymptotic expansions)

(1) $\forall \rho \in \mathbb{Q}$, $\quad \widetilde{\textcircled{H}}(\tau) \underset{\tau \to \rho}{\sim} \displaystyle\sum_{-\infty \ll j \in \frac{1}{2}\mathbb{Z}} {}^{\exists} c_j(\rho) \cdot (\tau - \rho)^j$.

(2) $\rho = \dfrac{h}{k}$, $\quad h \neq 0$ fixed

$$\Rightarrow c_j(\rho) \underset{k \to \infty}{\sim} \sum_{\theta \in {}^{\exists} S \subset \mathbb{Q}/\mathbb{Z}} e^{2\pi i \theta \cdot \frac{-1}{\rho}} \cdot {}^{\exists}\varphi_{\theta, h, j}(k).$$

S: fin, explicit

$\varphi_{\theta,h,j} \in \mathbb{C}((k^{-\frac{1}{2}}))$

Applying $\widetilde{\textcircled{H}}(\tau)$ as GPPV inv

$\rightsquigarrow$ Thm 1.

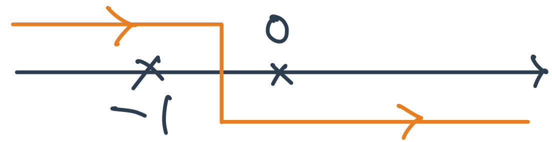
$$\begin{tikzcd}
Z_k \arrow[d, dashed, "k\to\infty" ', "?"] & & \hat{Z}(\tau) \arrow[ll, "\tau \to \frac{1}{k}" ', "①"] \arrow[d, "② \ \tau \mapsto -\frac{1}{\tau}"] \\
\sum e^{2\pi i \theta k} Z_\theta(k) & & {} \arrow[ll, "③" ', "\tau \to -k,\ k \to \infty"]
\end{tikzcd}$$

§4 Poisson summation formula with sign

Thm (YM)

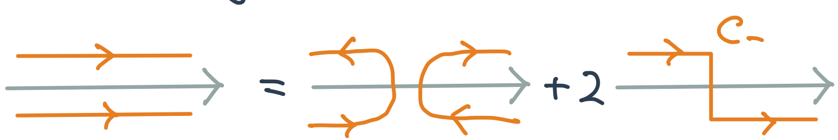
$$\sum_{m\in\mathbb{Z}} \mathrm{sgn}(m)\, f(m) = \sum_{n\in\mathbb{Z}} \mathrm{sgn}(n)\, \hat{f}(n) + \int_{C_-} \hat{f}(\xi)\, \frac{d\xi}{1-e^{2\pi i \xi}},$$

where $\mathrm{sgn}(0) := 1$, $\hat{f}(\xi)$: the Fourier trasform of f,

C_- : 

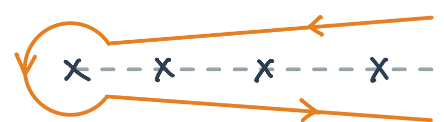
.

Rem • $\exists$ $\mathbb{Z}^r$ version. $\sum \int$ terms appear.

• Proof : LHS $\rightsquigarrow$ 

$\rightsquigarrow$ RHS

Fourier inverse formula

• Same strategy as resurgence : 

Sketch of a proof of Thm 2

False theta functions ($\supset$ GPPV invariants)

$$\widetilde{\Theta}(\tau) = \sum_{m\in\mathbb{Z}^r} \operatorname{sgn}(m)\, f(\tau;m)$$

$$= \sum_{n\in\mathbb{Z}^r} \operatorname{sgn}(n)\, \hat{f}(\tau;n) + \sum \int_{C_-} + \cdots + \int_{C_-^r}$$

(Poisson summation formula with sign)

$$= \sum_i \widetilde{\Theta}_i\left(-\frac{1}{\tau}\right)\cdot\tau^{\bullet} + \sum_j \widetilde{\Theta}_j\left(-\frac{1}{\tau}\right)\underline{\Omega_j(\tau)} + \cdots + \underline{\Omega(\tau)}$$

extend holomorphically to $\widetilde{\mathbb{C}\setminus\{0\}}$

$\rightsquigarrow$ Quantum modularity !!

§5 "Modular series"

Need to treat complicated functions (e.g. GPPV inv)

$\rightsquigarrow$ Estabish systematic framework

$\rightsquigarrow$ "Modular series": $\displaystyle\sum_{m\in\mathbb{Z}^r} g(m)\,\underline{\gamma(\tau;m)}$.

e.g. $q^{Q(m)}$

sgn × polynomials × periodic maps

We prove:

- Integral representation.
- Modular transformation.
- Asymptotic expansions. ($\supset$ stationary phase approximation)

Def Modular series is $\sum_{m\in\mathbb{Z}^r} g(m)\,\gamma(\tau;m)$, where

- $\gamma : \mathbb{H}\times\mathbb{R}^r \to \mathbb{C}$ s.t. $\begin{cases} \triangleright \text{ continuous,} \\ \triangleright \text{ hol in } \tau\in\mathbb{H}, \\ \triangleright \text{ exp decay in } x\in\mathbb{R}^r, \\ \triangleright\ \hat{\gamma}\in L^1(\mathbb{R}^r + i^{\forall}b). \end{cases}$

- $g \in \overline{\Omega} \cup \widetilde{\Omega}$,

  $\triangleright$ $\overline{\Omega} := \mathbb{Q}\left[x_i,\ \mathbb{1}_{a+N\mathbb{Z}}(x_i) \,\middle|\, \begin{matrix} 1\le i\le r, \\ a, N\in\mathbb{Z} \end{matrix}\right]$

  "quasi-polynomials" $\cap$

  $\triangleright$ $\widetilde{\Omega} := \sum_{\lambda\in\mathbb{Z}^r} \operatorname{sgn}(x-\lambda)\,\overline{\Omega} \quad \subset \quad \{g : \mathbb{Z}^r \to \mathbb{Q}\}$

  "false quasi-polynomials"

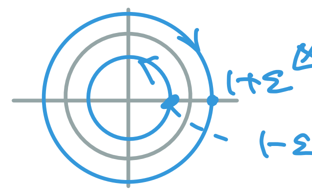
Key lemma

Def $$\mathfrak{R} := \mathbb{Q}\left[z_i^{\pm 1}, \frac{1}{1-z_i^N} \,\middle|\, 1\le i\le r,\ N\in\mathbb{Z}_{>0}\right]$$

"cyclotomic rational function"

Lem $\overline{\Omega} \cong \mathfrak{R} \cong \widetilde{\Omega}$ as $\mathbb{Q}$-vector spaces via

- $\mathfrak{R} \longrightarrow \overline{\Omega}$; $G(z) \longmapsto \left(m\in\mathbb{Z}^r \longmapsto 2^{-r}\int_{C^r} G(z)\prod_{1\le i\le r}\frac{z_i^{-m_i}\,dz_i}{2\pi\sqrt{-1}\,z_i}\right)$



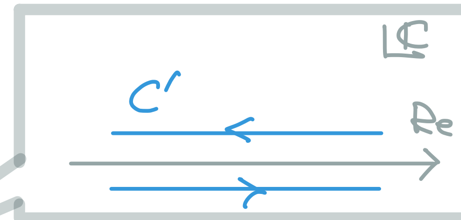
same as in def of GPPV inv!

- $\mathfrak{R} \longrightarrow \widetilde{\Omega}$; $G(z) \longmapsto \left(m\in\mathbb{Z}^r \longmapsto \mathrm{PV}\int_{|z_i|=1,\ 1\le i\le r} \quad '' \quad \right)$

Integral representation

Lem • $\gamma : \mathbb{H} \times \mathbb{R}^r \to \mathbb{C}$: as above (e.g. $q^{Q(x)}$)

• $$\begin{array}{ccccc} \overline{\Omega} & \simeq & \mathfrak{A} & \simeq & \widetilde{\Omega} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \overline{g}(x) & \longleftrightarrow & G(z) & \longleftrightarrow & \widetilde{g}(x) \end{array}$$



$\Longrightarrow$ • $$\sum_{m \in \mathbb{Z}^r} \overline{g}(m)\, \gamma(\tau; m) = 2^{-r} \int_{C'^r} G\left(\left(e^{2\pi i \xi_j}\right)_{1 \le j \le r}\right) \hat{\gamma}(\tau; \xi)\, d\xi$$

• $$\sum_{m \in \mathbb{Z}^r} \widetilde{g}(m)\, \gamma(\tau; m) = \mathrm{PV} \int_{\mathbb{R}^r} G\left(\left(e^{2\pi i \xi_j}\right)_{1 \le j \le r}\right) \hat{\gamma}(\tau; \xi)\, d\xi$$

(cf. median sum)

cf. Laplace integral in resurgence theory.

This lemma $\rightsquigarrow$ modular trasformation & asymptotic expansion.

Application: Integral representations of GPPV invariants

GPPV inv
$$\hat{Z}_b(q;M) := q^{\Delta} \sum_{\ell \in b + 2W(\mathbb{Z}^V)} \underbrace{F_\ell}_{g(\ell)} \underbrace{q^{-{}^t\ell W^{-1}\ell/4}}_{\gamma(\tau;\ell)},$$

where
$$\tilde{\Omega}_V \ni (\ell \mapsto F_\ell) \underset{PV\int_{|z|=1}}{\longleftrightarrow} F(z) := \prod_{v\in V} (z_v - z_v^{-1})^{2-\deg(v)} \in \mathcal{S}_d\hat{V}$$

$\rightsquigarrow$

$$\boxed{\begin{gathered} \forall \alpha \in H^1(M,\mathbb{Z}), \\ \sum_{b\in \mathrm{Spin}^c(M)} e^{2\pi i\, \mathrm{lk}(\alpha,b)}\, \hat{Z}_b(q;M) \\ = \sqrt{\frac{i}{\tau}}^{|V|} \frac{q^{\Delta}}{\sqrt{|H_1(M,\mathbb{Z})|}} PV\int_{\mathbb{R}^V} \tilde{q}^{-2\,{}^t\xi W \xi}\, F\left(\left(e^{2\pi i \xi_v}\right)_{v\in V}\right) d\xi. \end{gathered}}$$

Conclusion

① Asymptotic expansions of WRT invariants

② Quantum modularity / asymptotic expansions for False theta functions ($\supset$ GPPV invariants)

③ New techniques :

- Poisson summation formula with sign
- Modular series

Thank you!