Effective cones of moduli spaces of stable rational curves

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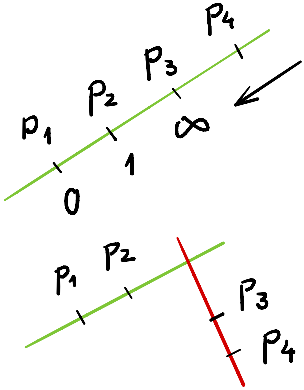
June 2021

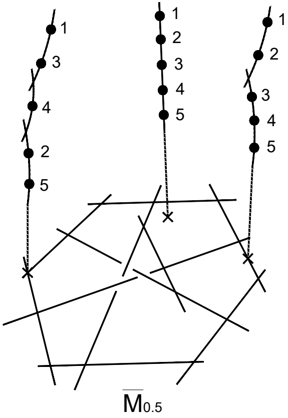
Richmond Geometry Festival

Moduli spaces of stable rational curves

- $M_{0,n} = \left\{ \substack{p_1,\ldots,p_n \in \mathbb{P}^1 \\ p_i \neq p_j} \right\} / \mathrm{PGL}_2$
- $M_{0,3} = \mathrm{pt}$ (send $p_1, p_2, p_3 \to 0, 1, \infty$)
- $M_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ (via cross-ratio)
- $\overline{M}_{0,4} = \mathbb{P}^1$
- $\overline{M}_{0,n}$ functorial compactification



The moduli space $\overline{\mathrm{M}}_{0,n}$



A pointed rational stable curve $(C, p_1, \ldots, p_n)$:

- an at worst nodal curve C with $p_a(C) = 0$
- smooth, distinct points $p_1, \ldots, p_n$,
- $\omega_C(p_1 + \ldots + p_n)$ is ample

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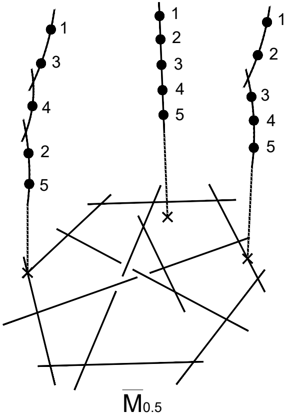

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$\omega_{C|C'} = \omega_{C'}(\text{nodes on } C') = \mathcal{O}(-2)(\#\text{nodes})$

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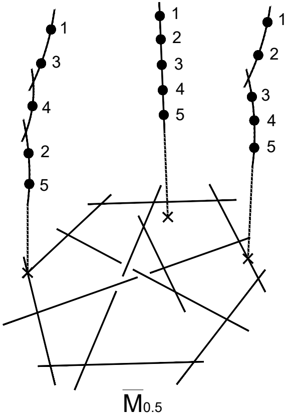

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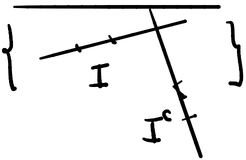
Boundary divisors

$I \sqcup I^c = \{1, \ldots, n\}$

$|I|, |I^c| \geq 2$



What kind of variety is $\overline{\mathrm{M}}_{0,n}$?

- (Kapranov models) $\overline{\mathrm{M}}_{0,n} = \ldots \mathrm{Bl}_{\binom{n-1}{3}} \mathrm{Bl}_{\binom{n-1}{2}} \mathrm{Bl}_{n-1} \mathbb{P}^{n-3}$

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- $\overline{\mathrm{M}}_{0,n}$, $n \geq 8$: $-K$ not pseudo-effective

Effective cones

X normal projective variety

$\mathrm{N}^1(X) = \mathrm{Pic}(X)/_{\equiv} \quad (D \equiv D' \Leftrightarrow D \cdot C = D' \cdot C$ for all curves $C \subseteq X)$

$\overline{\mathrm{Eff}}(X) = \overline{\{\sum a_i D_i \ | a_i \in \mathbb{Q}_{\geq 0}, \ D_i \ \text{ effective } \}} \subseteq \mathrm{N}^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$

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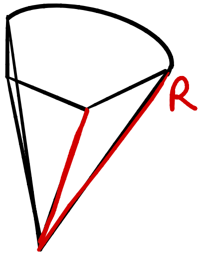

$\mathcal{C} \subseteq \mathbb{R}^n$ cone

A ray $R = \mathbb{R}_{\geq 0}\{v\}$ of $\mathcal{C}$ is extremal if

$v = v_1 + v_2,\ v_1, v_2 \in \mathcal{C} \Longrightarrow v_1, v_2 \in R$

The effective cone of $\overline{\mathrm{M}}_{0,n}$

- Every boundary divisor is contracted by a Kapranov map $\overline{\mathrm{M}}_{0,n} \to \mathbb{P}^{n-3}$ and generates an extremal ray of $\overline{\mathrm{Eff}}(\overline{\mathrm{M}}_{0,n})$

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- More extremal divisors for $n \geq 7$ (Opie 2016, based on Chen–Coskun 2014, Doran–Giansiracusa–Jensen 2017)

The effective cone of $\overline{\mathrm{M}}_{0,n}$

Theorem (C.–Laface–Tevelev–Ugaglia 2020)

The cone $\overline{Eff}(\overline{M}_{0,n})$ is not polyhedral for $n \geq 10$, both in characteristic 0 and in characteristic p, for an infinite set of primes p of positive density (including all primes up to 2000).

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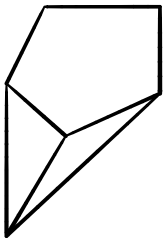
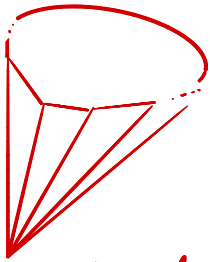


polyhedral cone



non polyhedral cone

Rational contractions

Definition

A rational contraction $X \dashrightarrow Y$ between $\mathbb{Q}$-factorial, normal projective varieties, is a rational map that can be decomposed into a sequence of

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Theorem

Let $X \dashrightarrow Y$ be a rational contraction. If X has any of these properties then Y does as well:

- *Mori Dream Space (Keel–Hu 2000, Okawa 2016)*
- *(rational) polyhedral effective cone (BDPP 2013)*

$\overline{\mathrm{M}}_{0,n}$ and blow-ups of toric varieties

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Theorem (C.–Tevelev 2015)

There are rational contractions

$$Bl_e\,\overline{LM}_{0,n+1} \dashrightarrow \overline{M}_{0,n} \to Bl_e\,\overline{LM}_{0,n},$$

where $\overline{LM}_{0,n}$ is the Losev-Manin moduli space of dimension $n-3$, $e =$ identity point of the open torus $\mathbb{G}_m^{n-3} \subseteq \overline{LM}_{0,n}$.

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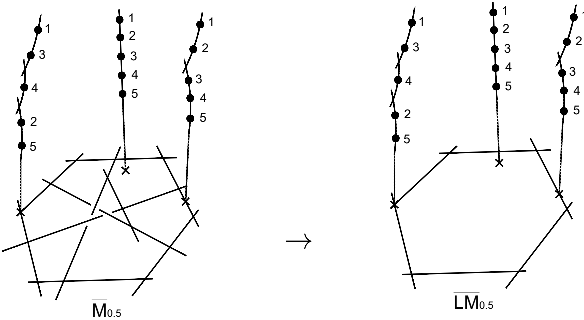
Kapranov description: $\overline{\mathrm{LM}}_{0,n} = \ldots \mathrm{Bl}_{\binom{n-2}{3}} \mathrm{Bl}_{\binom{n-2}{2}} \mathrm{Bl}_{n-2}\,\mathbb{P}^{n-3}$
(blow-up $n-2$ points, all lines, planes,... spanned by them)

THE LOSEV-MANIN MODULI SPACE $\overline{\mathrm{LM}}_{0,n}$

The Losev-Manin moduli space $\overline{\mathrm{LM}}_{0,n}$ is the Hassett moduli space of stable rational curves with n markings and weights $1, 1, \epsilon, \ldots, \epsilon$.



trees of $\mathbb{P}^1$'s

chains of $\mathbb{P}^1$'s

Universal blown up toric variety

Theorem

X projective $\mathbb{Q}$-factorial toric variety. For $n \gg 0$

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Corollary (C.–Tevelev, 2015)

$\overline{M}_{0,n}$ is not a MDS in characteristic 0 for $n \gg 0$. There exists a rational contraction

$$\overline{M}_{0,n} \dashrightarrow Bl_e \mathbb{P}(a, b, c)$$

for some a, b, c such that $Bl_e \mathbb{P}(a, b, c)$ has a nef but not semi-ample divisor (Goto–Nishida–Watanabe 1994).

Remark

This argument cannot work in characteristic p, where, by Artin's contractibility criterion, a nef divisor on $Bl_e \mathbb{P}(a, b, c)$ is semi-ample.

Blown up toric surfaces

Theorem (C.-Laface-Tevelev-Ugaglia 2020)

There exist projective toric surfaces $\mathbb{P}_\Delta$, given by good polygons Δ, such that $\overline{Eff}(Bl_e\,\mathbb{P}_\Delta)$ is not polyhedral in characteristic 0.

For some of these toric surfaces, $\overline{Eff}(Bl_e\,\mathbb{P}_\Delta)$ is not polyhedral in characteristic p for an infinite set of primes p of positive density.

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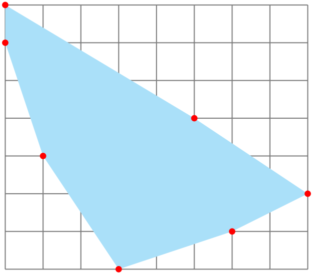
Corollary

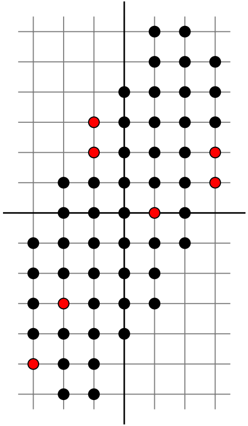
For $n \geq 10$, the space $\overline{M}_{0,n}$ is not a MDS both in characteristic 0 and in characteristic p for an infinite set of primes of positive density, including all primes up to 2000.

Example of a good polygon



Example of a good polygon

There is a rational contraction $\overline{\mathrm{M}}_{0,10} \to \mathrm{Bl}_e\, \overline{\mathrm{LM}}_{0,10} \dashrightarrow \mathrm{Bl}_e\, \mathbb{P}_\Delta$:



Red $\to$ normal fan of Δ Black $\to$ projection of fan of $\overline{\mathrm{LM}}_{0,10}$

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$C^\perp \subseteq \mathrm{Cl}(X)$ orthogonal complement of C ; $C^\perp$ contains C

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If X, C are defined over a finite field, then $e(C, X) < \infty$.

Non-polyhedrality criterion

Lemma

- *If* $e = e(C, X) < \infty$, *then* $h^0(eC) = 2$ *and* $|eC| : X \to \mathbb{P}^1$ *is an elliptic fibration with* C *a multiple fiber.*
- *If* $e(C, X) = \infty$, *then* C *is rigid :*

$$h^0(nC) = 1 \quad \text{for all} \quad n \geq 1.$$

In this case, $\overline{Eff}(X)$ *is not polyhedral if* $\rho(X) \geq 3$*.*

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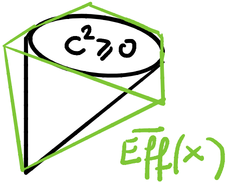

Nikulin: $\rho(X) \geq 3$, $\overline{\mathrm{Eff}}(X)$ is polyhedral $\Rightarrow$ every irreducible curve C with $C^2 = 0$ is contained in the interior of a facet; in particular, a multiple moves.

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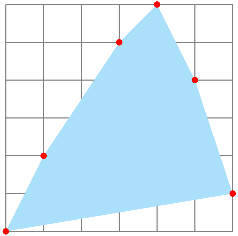
(I) *The Newton polygon of C coincides with Δ ($\Leftrightarrow C \subseteq X^{smooth}$),*

(II) *$Vol(\Delta) = m^2$ and $|\partial\Delta \cap \mathbb{Z}^2| = m$ ($\Leftrightarrow C^2 = 0$, $p_a(C) = 1$),*

(III) *The restriction $\mathrm{res}(C) = \mathcal{O}_X(C)|_C$ is not torsion in $\mathrm{Pic}^0(C)$ over $\mathbb{Q}$.*

Example

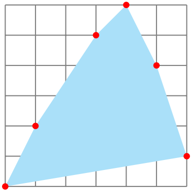


$$\mathrm{Vol}(\Delta) = 36, \quad |\partial\Delta \cap \mathbb{Z}^2| = 6$$

The linear system $|\mathcal{L}_\Delta - 6E|$ contains a unique curve C with equation

$$x^4y^6 + 6x^5y^4 - 2x^4y^5 - 14x^5y^3 - 17x^4y^4 - 4x^3y^5 +$$
$$+x^6y + 11x^5y^2 + 38x^4y^3 + 26x^3y^4 - 9x^5y - 27x^4y^2 -$$
$$-34x^3y^3 + 22x^4y + 16x^3y^2 - 10x^2y^3 - 24x^3y +$$
$$+10x^2y^2 + 15x^2y + 5xy^2 - 11xy + 1 = 0.$$

Example



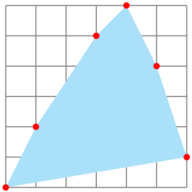
The curve C is a smooth elliptic curve labelled 997.a1 in the LMFDB database. It has the minimal equation

$$y^2 + y = x^3 - x^2 - 24x + 54$$

The Mordell-Weil group $C(\mathbb{Q})$ is $\mathbb{Z} \times \mathbb{Z}$, with generators

$$Q = (1, 5), \quad P = (6, -10)$$

Example



The curve C is a smooth elliptic curve labelled 997.a1 in the LMFDB database. It has the minimal equation

$$y^2 + y = x^3 - x^2 - 24x + 54$$

The Mordell-Weil group $C(\mathbb{Q})$ is $\mathbb{Z} \times \mathbb{Z}$, with generators

$$Q = (1,5), \quad P = (6,-10)$$

Computation : $\mathrm{res}(C) = -Q$ (not torsion, so Δ is good)

Minimal elliptic pairs

Polyhedrality when $e(C,X) < \infty$? In general, for any $e(C,X)$:

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$(K+C)\cdot C=0$, C is nef $\implies$ If $(K+C)\cdot E<0$ then

E is a component of $K+C$ and $C\cdot E=0$, $K\cdot E<0$.

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Run $(K + C)$-MMP (Tanaka 2014, Fujino 2020). □

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Remark: (C, X) is minimal, $\mathcal{O}(C)_{|C}$ not torsion, $\mathcal{O}(K + C)_{|C} = \mathcal{O}_{|C}$ $\Rightarrow$

$$K + C \sim 0$$

In particular, K is a Cartier divisor and X has Du Val singularities.

Minimal + Du Val singularities

Definition

Since $K \cdot C = 0$, *define on* $\mathrm{Cl}_0(X) = C^{\perp}/\langle K \rangle$ *the reduced restriction map*

$$\overline{\mathrm{res}} : \mathrm{Cl}_0(X) \to \mathrm{Pic}^0(C)/\langle \mathrm{res}(K) \rangle$$

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Let (C, Y) *be an elliptic pair such that* Y *has Du Val singularities. Let* Z *be the minimal resolution of* Y*. Then*

$$(C, Y) \quad \textit{minimal} \quad \Leftrightarrow \quad (C, Z) \quad \textit{minimal} \quad \Leftrightarrow \quad \rho(Z) = 10.$$

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Assume (C, Y) minimal elliptic pair with $e(C, Y) < \infty$. Then

$$\overline{\mathit{Eff}}(Y) \quad \textit{polyhedral} \quad \Leftrightarrow \quad \overline{\mathit{Eff}}(Z) \quad \textit{polyhedral} \quad \Leftrightarrow$$

$$\mathrm{Ker}(\overline{\mathrm{res}}) \quad \textit{contains 8 linearly independent roots of} \quad \mathbb{E}_8.$$

Upshot

$(C, Y) =$ minimal model of elliptic pair (C, X)

- $e(C, X) = \infty \Rightarrow \overline{\mathrm{Eff}}(X)$, $\overline{\mathrm{Eff}}(Y)$ not polyhedral (if $\rho \geq 3$)

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Problem

- *Suppose* C, X *are defined over* $\mathbb{Z}$, $e(C, X) = \infty$
- $X \to Y$ *extends to the morphism of integral models* $\mathcal{X} \to \mathcal{Y}$ *over* $\operatorname{Spec} \mathbb{Z}$ *(outside of finitely many primes of bad reduction)*
- (C_p, Y_p) *is still the minimal elliptic pair associated to* (C_p, X_p)
- $e(C_p, X_p) < \infty$*. Study distribution of "polyhedral" primes*

Example - Minimal elliptic pair

(C, X) elliptic pair, $X = \mathrm{Bl}_e \mathbb{P}_\Delta$

Zariski decomposition $K_X + C = N + P$, $N = 3C_1 + 2C_2$, $P = 0$

To get minimal elliptic pair (C, Y), contract C_1, C_2.

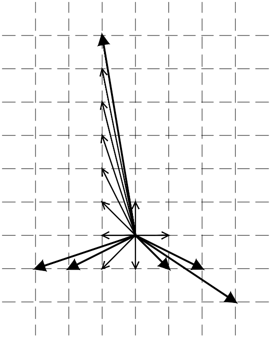
$$\begin{array}{ccc} \mathrm{Bl}_e \tilde{\mathbb{P}}_\Delta & \longrightarrow & Z \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

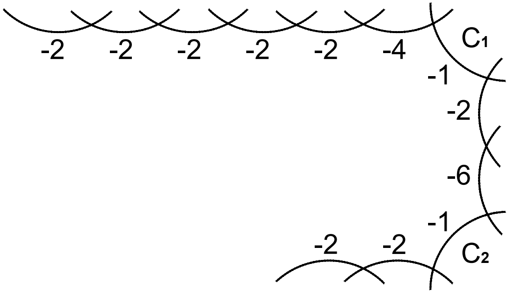
$Z \to Y$ minimal resolution, $\rho(X) = 5$, $\rho(Y) = 3$, $\rho(Z) = 10$

$\tilde{\mathbb{P}}_\Delta$ is the minimal resolution of $\mathbb{P}_\Delta$

Example - Minimal resolution

Fan of the minimal resolution $\tilde{\mathbb{P}}_\Delta$ of $\mathbb{P}_\Delta$:





C_1, C_2 = proper transforms of 1-parameter subgroups $\{v = 1\}$, $\{u = 1\}$

Example - Minimal resolution

$Z \to Y$ minimal resolution of Y, $\mathsf{Cl}(Z) = \mathsf{Cl}(Y) \oplus T$

$T =$ sublattice spanned by classes of (-2) curves above singularities of Y

T is contained in the kernel of the reduced restriction map

$$\overline{\mathsf{res}} : \mathbb{E}_8 = \mathsf{Cl}_0(Z) \to \mathsf{Pic}^0(C)/\langle Q \rangle, \quad Q = (1, 5)$$

Computations:

- $T = \mathbb{A}_7$
- Images of roots of $\mathbb{E}_8$ in $\mathbb{E}_8/\mathbb{A}_7 = \mathbb{Z}\{\alpha\}$ are $\pm k\alpha$, for $0 \leq k \leq 3$
- $\mathsf{res}(\alpha) = P - Q$ where $P = (6, -10)$

$\overline{\mathsf{Eff}}(Y)$ not polyhedral in characteristic $p \Leftrightarrow k\overline{P} \notin \langle \overline{Q} \rangle$ for $k = 1, 2, 3$

Example - Non-polyhedral primes

Prove that the set of primes p such that

$$\overline{P}, 2\overline{P}, 3\overline{P} \notin \langle \overline{Q} \rangle \subseteq C(\mathbb{F}_p)$$

has positive density.

Fix q prime. It suffices to prove that the set of primes p such that

- q divides the index of $\langle \overline{Q} \rangle \subseteq C(\mathbb{F}_p)$
- q does not divide the index of $\langle 6\overline{P} \rangle \subseteq C(\mathbb{F}_p)$

has positive density.

Apply Chebotarev's Density theorem + a theorem of Lang-Trotter

Non-polyhedral primes

The set of non-polyhedral primes $p < 2000$ for the previous example of a good polygon:

$$7, 11, 41, 67, 173, 307, 317, 347, 467, 503, 523, 571, 593, 631, 677, 733,$$
$$797, 809, 811, 827, 907, 937, 1019, 1021, 1087, 1097, 1109, 1213, 1231,$$
$$1237, 1259, 1409, 1433, 1439, 1471, 1483, 1493, 1567, 1601, 1619, 1669,$$
$$1709, 1801, 1811, 1823, 1867, 1877, 1933, 1951, 1993$$

This gives 18% of the primes under 2000.

Thank you!