Volumes and Intersection Theory on Moduli Spaces of Differentials

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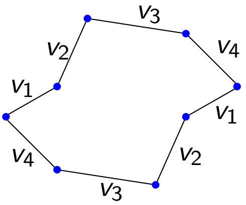
- Let ω be a *holomorphic differential* on a Riemann surface X.
- Away from the zeros, $\omega = dz = dx + idy \implies \omega$ induces a flat metric on $X \setminus (\omega)_0$.
- At a zero of order m, $\omega = d(w^{m+1}) \implies$ it corresponds to a saddle point of angle $2\pi(m+1)$.

Example (A differential with a double zero on a genus two surface)



- All vertices are identified as the saddle point of angle 6π.

Moduli spaces of holomorphic differentials

- Let $\mu = (m_1, \ldots, m_n)$ be a partition of $2g-2$. Define the *moduli space of holomorphic differentials of type* μ by

$$\mathcal{H}(\mu) = \Big\{ (X, \omega) \mid X \text{ is a Riemann surface of genus } g,$$

$$\omega \text{ is a holomorphic differential with } (\omega)_0 = m_1 z_1 + \cdots + m_n z_n \Big\}.$$

Period coordinates of $\mathcal{H}(\mu)$

- Let $\gamma_1, \ldots, \gamma_{2g}; \gamma_{2g+1}, \ldots, \gamma_{2g+n-1}$ be a basis of the relative homology group $H_1(X, z_1, \ldots, z_n; \mathbb{Z})$.
- $$\left(\int_{\gamma_1} \omega, \ldots, \int_{\gamma_{2g+n-1}} \omega\right)$$
  provides a local coordinate system of $\mathcal{H}(m_1, \ldots, m_n)$ at (X, ω), called *period coordinates*.
- Period coordinates correspond to the "edges" of the polygon representation of (X, ω).
- $\mathcal{H}(m_1, \ldots, m_n)$ is a complex orbifold of dimension $2g + n - 1$.

Masur-Veech volumes

- Let $\mathcal{H}_1(\mu) \subset \mathcal{H}(\mu)$ be the hypersurface parameterizing (X, ω) of area one.
- Period coordinates of $\mathcal{H}(\mu)$ induces a natural Lebesgue measure on $\mathcal{H}_1(\mu)$.
- [Masur; Veech, 1982] showed that the corresponding volume of $\mathcal{H}_1(\mu)$ for any μ is finite, called the *Masur-Veech volume*.

- Recall that ψ_i is the cotangent line bundle class associated with the i-th marked point.
- Let η be the tautological line bundle class $\mathcal{O}(-1)$ of $\mathbb{P}\overline{\mathcal{H}}$, where $\mathcal{O}(-1)|_{(X,\omega)} = \mathbb{C} \cdot \omega$.

Theorem (C.-Möller-Sauvaget-Zagier, 2019)

For all partitions $\mu = (m_1, \ldots, m_n)$ of $2g-2$, the Masur-Veech volume of $\mathcal{H}(\mu)$ equals the intersection number

$$\mathrm{Vol}(\mathcal{H}(\mu)) = \frac{-2(2\pi i)^{2g}}{(2g-3+n)!} \cdot \frac{1}{m_1+1} \int_{\mathbb{P}\overline{\mathcal{H}}_{g,n}(\mu)} \eta^{2g-1} \psi_2 \cdots \psi_n.$$

A heuristic viewpoint via hermitian metrics

- The Masur-Veech volume form on $\mathbb{P}\mathcal{H}(m_1, \ldots, m_n)$ is of type "Abs $\wedge$ Rel":

$$\left(\prod_{i=2}^{2g} (da_i \wedge d\bar{a}_i) \right) \wedge \prod_{i=2}^{n} (dr_i \wedge d\bar{r}_i),$$

where the a_i and r_i correspond to the absolute and relative period coordinates respectively.

- The line bundle $\mathcal{O}_{\mathbb{P}\mathcal{H}}(-1)$ carries a hermitian metric induced by the area form:

$$h(X, \omega) = \text{Area}(X, \omega) = \frac{i}{2} \sum_{i=1}^{g} (a_i \bar{a}_{g+i} - a_{g+i} \bar{a}_i).$$

- η^{2g-1} corresponds to the Abs part $\bigwedge^{2g-1}(\partial\bar{\partial} \log h)$.

- One expects to find hermitian metrics on the ψ-bundles such that $\psi_2 \cdots \psi_n$ corresponds to the Rel part.
- It remains to justify the singular loci and extensions to the boundary of $\mathbb{P}\overline{\mathcal{H}}_{g,n}(\mu)$ for these hermitian metrics.

An actual proof via recursions

- The intersection numbers satisfy a recursion by merging zeros.
- The volumes satisfy a recursion by counting torus covers.
- The two recursions are equivalent and agree on the minimal space $\mathcal{H}(2g-2)$.
- It implies that they are equal for all $\mathcal{H}(\mu)$.

Volume asymptotics

- Eskin-Zorich conjectured that $(m_1+1)\cdots(m_n+1)\,\mathrm{Vol}(\mathcal{H}(\mu)) \to 4$ as $g \to \infty$.
- [Aggarwal, 2018] proved it by a combinatorial argument.

Theorem (CMSZ)

The volume-intersection formula can determine the volume asymptotic:

$$(m_1+1)\cdots(m_n+1)\,\mathrm{Vol}(\mathcal{H}(\mu)) \sim 4 - \frac{2\pi^2}{3(2g-2+n)} + O\left(\frac{1}{g^2}\right).$$