The Kodaira dimension of $\overline{\mathcal{M}}_g$: latest progress on a century-old problem

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A classical problem

Theorem

(Severi 1915) The moduli space $\mathcal{M}_g$ of curves of genus g is unirational for $g \leq 10$.

- Severi used in his proof plane curves of minimal degree whose nodes are in general position. The result implies that one can write down explicitly the general curve of genus g in a family depending on free parameters.
- Severi's unirationality result predates the proof of the existence of $\mathcal{M}_g$!
- Sernesi, Chang-Ran (1980s), Verra (2005): $\mathcal{M}_g$ is unirational for $g = 11, 12, 13, 14$. For all $g \leq 14$ Schreyer has produced efficient programs writing down the random curve $C/\mathbb{F}_q$ of genus g.
- Chang-Ran (1987), Bruno-Verra (2005), Schreyer (2016): $\overline{\mathcal{M}}_{15}$ is rationally connected.
- Interesting news on $\mathcal{M}_{16}$. Discussing it would take us too far afield.

Theorem

(Harris, Mumford, Eisenbud 1982-87) $\overline{\mathcal{M}}_g$ is of general type for $g \geq 24$.

The Harris-Mumford approach

• Indirectly inspired by the work of Freitag and Tai on $\mathcal{A}_g$.

$$\overline{\mathcal{M}}_g \setminus \mathcal{M}_g = \Delta_0 \cup \Delta_1 \cup \ldots \cup \Delta_{\lfloor \frac{g}{2} \rfloor},$$

where Δ_i are the irreducible boundary divisors in $\overline{\mathcal{M}}_g$. Precisely,
$\Delta_0 := \{[C/p \sim q] : C$ of genus $g-1$ and $p, q \in C\}^-$ and for $i \geq 1$
$\Delta_i := \{[C_1 \cup C_2] : C_1$ of genus i, C_2 of genus $g-i\}^-$

• The Hodge class: $\lambda := [\mathbb{E}] \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$, where $\mathbb{E} \to \overline{\mathcal{M}}_g$ is the Hodge bundle with fibres $\mathbb{E}[C] = \bigwedge^g H^0(C, \omega_C)$, for any stable curve C.

Theorem

(Harer, Arbarello-Cornalba) For $g \geq 3$, the group $CH^1(\overline{\mathcal{M}}_g)$ is freely generated by the classes $\lambda, \delta_0, \ldots, \delta_{\lfloor g/2 \rfloor}$.

• Via a Riemann-Roch calculation on the universal curve, Harris and Mumford computed the canonical class of the moduli space:

$$K_{\overline{\mathcal{M}}_g} = 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \cdots - 2\delta_{\lfloor \frac{g}{2} \rfloor}.$$

Since the singularities of $\overline{\mathcal{M}}_g$ do not impose adjunction conditions (Harris-Mumford), $\overline{\mathcal{M}}_g$ is of general type if and only if $K_{\overline{\mathcal{M}}_g}$ is big.

Strategy: Find an effective divisor $D \subseteq \overline{\mathcal{M}}_g$ such that $[D] = a\lambda - \sum_{i=0}^{\lfloor g/2 \rfloor} b_i \delta_i$, with $a, b_i \geq 0$ and its slope

$$s(D) := \frac{a}{\min_{i \geq 0} b_i} < s(K_{\overline{\mathcal{M}}_g}) = \frac{13}{2}.$$

Then for $\alpha, \beta > 0$ we can write that

$$K_{\overline{\mathcal{M}}_g} = \alpha \cdot \lambda + \beta \cdot D + \{\text{non-negative combination of } \delta_i\}.$$

Since λ is big (its sections correspond to Siegel modular forms), it follows that $K_{\overline{\mathcal{M}}_g}$ is big, that is, $\overline{\mathcal{M}}_g$ is of general type.

Summary: $\overline{\mathcal{M}}_g$ of general type $\Leftrightarrow$ there exists an effective divisor $D \subseteq \overline{\mathcal{M}}_g$ with $s(D) < \frac{13}{2}$.

• D can be chosen to be a geometric divisor, i.e. points of D can be characterized by a geometric property (existence of a linear system, a special syzygy, a Hurwitz condition ...)

- Brill-Noether Theorem: If C is a general curve of genus g, the variety

$$W^r_d(C) := \{L \in \mathrm{Pic}^d(C) : h^0(C,L) \geq r+1\}$$

has dimension $\rho(g,r,d) = g - (r+1)(g-d+r)$.

Theorem

(Eisenbud-Harris 1987) If $\rho(g,r,d) = -1$, the locus $\mathcal{M}^r_{g,d} := \{[C] \in \mathcal{M}_g : W^r_d(C) \neq \emptyset\}$ is a divisor in $\mathcal{M}_g$. The class of its closure in $\overline{\mathcal{M}}_g$ is $[\overline{\mathcal{M}}^r_{g,d}] = c\Big((g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i=1}^{\lfloor g/2 \rfloor} i(g-i)\delta_i\Big)$.

Thus $s(\overline{\mathcal{M}}^r_{g,d}) = 6 + \frac{12}{g+1} < \frac{13}{2}$ for $g \geq 24$. This proves the Harris-Mumford-Eisenbud theorem (at least when $g+1$ is composite).

- Can one construct divisors of slope $< 6 + \frac{12}{g+1}$ (Slope Conjecture)?

Theorem

(Farkas-Popa 2003) If $D \subseteq \overline{\mathcal{M}}_g$ is an effective divisor with $s(D) < 6 + \frac{12}{g+1}$, then

$$D \supseteq \mathcal{K}_g := \{[C] \in \mathcal{M}_g : C \text{ lies on a } K3 \text{ surface}\}.$$

The Strong Maximal Rank Conjecture

Fix g, r and d with $\rho(g,r,d) \geq 0$. Set $\sigma\colon \mathcal{G}^r_d \twoheadrightarrow \mathcal{M}_g$, with

$$\mathcal{G}^r_d := \big\{[C,L] : [C] \in \mathcal{M}_g \text{ and } L \in W^r_d(C)\big\}.$$

Maximal Rank Conjecture: For a general $[C,L] \in \mathcal{G}^r_d$, the maps

$$\phi^k_L\colon \mathrm{Sym}^k H^0(C,L) \to H^0(C,L^k)$$

given by multiplication of sections are of maximal rank for every $k \geq 2$. This is now a theorem of Eric Larson (2018), after a lot of previous work (Hirschowitz, Ballico-Ellia, Voisin, Jensen-Payne, Osserman ...)

- In the statement of MRC both C and L are general.

Strong Maximal Rank Conjecture (Aprodu-Farkas 2008) For a general curve C of genus g, the locus

$$\Sigma^r_d(C) := \big\{L \in W^r_d(C) : \mathrm{Sym}^2 H^0(L) \xrightarrow{\phi_L} H^0(L^2) \text{ not of maximal rank}\big\}$$

has the expected dimension, that is, $\rho(g,r,d) - (2d+1-g) + \frac{r(r+3)}{2}$.

• If $\rho(g,r,d) - (2d+1-g) + \frac{r(r+3)}{2} < 0$, we expect $\phi_L \colon \mathrm{Sym}^2 H^0(C,L) \to H^0(C,L^2)$ to be injective for all $L \in W^r_d(C)$. Note: $h^0(C,L^2) = 2d+1-g$, because $h^1(C,L^2) = 0$.

• For $\rho(g,r,d) = 0$, the MRC and the Strong MRC are equivalent (monodromy!) For $\rho(g,r,d) > 0$ the Strong MRC is incomparably more difficult. One cannot use projective degenerations! Powerful tropical methods have led to a breakthrough on the Strong MRC, with direct applications to the Kodaira dimension of $\overline{\mathcal{M}}_g$.

Theorem

(Farkas-Jensen-Payne 2020) Both moduli spaces $\overline{\mathcal{M}}_{22}$ and $\overline{\mathcal{M}}_{23}$ are of general type.

Divisors on $\overline{\mathcal{M}}_{22}$ and $\overline{\mathcal{M}}_{23}$ via Strong MRC

Fix $r = 6$ and $d = g + 3$. $\begin{cases} g = 22, d = 25 : C \xrightarrow{|L|} \mathbb{P}^6, \ \rho = 1 \\ g = 23, d = 26 : C \xrightarrow{|L|} \mathbb{P}^6, \ \rho = 2 \end{cases}$

The degeneracy locus of those $[C, L] \in \mathcal{G}^6_{g+3}$ for which the map

$$\phi_L \colon \mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^2)$$

is not injective has expected codimension $g - 20$, that is, 2 if $g = 22$ respectively 3 if $g = 23$. Consequently

$$\mathcal{D}_g := \big\{[C] \in \mathcal{M}_g : \exists L \in W^6_{g+3}(C) \text{ with } \phi_L \text{ not injective}\big\}$$

is a virtual divisor on $\mathcal{M}_g$. Since $\mathcal{K}_g \subseteq \mathcal{D}_g$ (sections of $K3$ surfaces fail Strong MRC!), $\mathcal{D}_g$ is a prime candidates to show that $\overline{\mathcal{M}}_g$ is of general type.

- Issues: (i) Transversality and (ii) compactifying $\mathcal{D}_g$.

Virtual classes of MRC divisors

• We choose a partial compactification $\widetilde{\mathcal{M}}_g$ of $\mathcal{M}_g$ differing from $\mathcal{M}_g \cup \Delta_0 \cup \Delta_1$ only in codimension 2 and a proper extension

$$\sigma\colon \widetilde{\mathcal{G}}^6_{g+3} \to \widetilde{\mathcal{M}}_g,$$

from the stack of limit linear series of type $\mathfrak{g}^6_{g+3}$.

• We construct locally free sheaves $\mathcal{E}$ and $\mathcal{F}$ over $\widetilde{\mathcal{G}}^6_{g+3}$ with $\mathsf{rk}(\mathcal{E}) = 7$ and $\mathsf{rk}(\mathcal{F}) = g + 7 (= 2d + 1 - g)$, together with a morphism

$$\phi\colon \mathsf{Sym}^2\mathcal{E} \to \mathcal{F},$$

s.t. for $[C, L] \in \mathcal{G}^6_{g+3}$, we have $\mathcal{E}(C, L) = H^0(L)$, $\mathcal{F}(L) = H^0(L^2)$ and

$$\phi_{[C,L]}\colon \mathsf{Sym}^2 H^0(C, L) \to H^0(C, L^2)$$

is the usual multiplication of sections.

Definition

Define the virtual class $[\widetilde{\mathcal{D}}_g]^{\text{vir}} := \sigma_*\left(c_{g-20}\big(\mathcal{F} - \mathsf{Sym}^2\mathcal{E}\big)\right) \in CH^1(\widetilde{\mathcal{M}}_g)$.

• If the degeneracy locus $\mathcal{U}$ of ϕ: $\mathrm{Sym}^2\mathcal{E} \to \mathcal{F}$ has the expected codimension 2 for $g = 22$, respectively 3 for $g = 23$, then $[\widetilde{\mathcal{D}}_g]^{\mathrm{vir}} = [\widetilde{\mathcal{D}}_g]$.

Theorem

(FJP) The virtual classes of the MRC divisors have the following slopes:

$$s\big([\widetilde{\mathcal{D}}_{22}]^{\mathrm{vir}}\big) = \frac{17121}{2636} = 6.495\ldots \text{ and } s\big([\widetilde{\mathcal{D}}_{23}]^{\mathrm{vir}}\big) = \frac{470749}{72725} = 6.473\ldots.$$

Conclusion: Both virtual slopes are $< \frac{13}{2}$! Transversality issues:

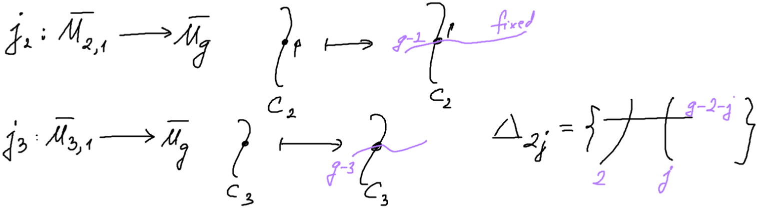
Theorem

(FJP) The Strong Maximal Rank Conjecture holds on $\overline{\mathcal{M}}_{22}$ and $\overline{\mathcal{M}}_{23}$. For a general C, the multiplication map ϕ_L is injective for every $L \in W^6_{g+3}(C)$.

• There is an alternative proof of this: Liu-Osserman-Teixidor-Zhang.
• This shows that $\sigma(\mathcal{U}) \neq \widetilde{\mathcal{M}}_g$ (recall $\sigma\colon \widetilde{\mathcal{G}}^6_{g+3} \to \widetilde{\mathcal{M}}_g$). To conclude $[\widetilde{\mathcal{D}}_g]^{\mathrm{virt}} = [\widetilde{\mathcal{D}}_g]$, one needs to rule out the possibility of a component $\mathcal{U}' \subseteq \mathcal{U}$ of excessive dimension, mapping via σ with positive dimensional fibres onto a divisor in $\widetilde{\mathcal{M}}_g$.

(i) For the general curve $[C] \in \mathcal{M}_g$ the multiplication map $\phi_L\colon \mathrm{Sym}^2 H^0(C,L) \to H^0(C,L^2)$ is injective for all $L \in W^6_{g+3}(C)$.

(ii) There exists no divisor $\mathcal{Z}$ in $\overline{\mathcal{M}}_g$ such that for each $[C] \in \mathcal{Z}$ the map ϕ_L is non-injective for infinitely many $L \in W^6_{g+3}(C)$.



To prove (ii) it suffices to verify all of the following:

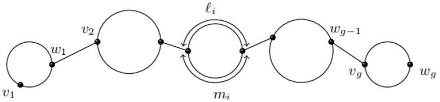
- $j_2^*(\mathcal{Z}) = 0$ and $j_3^*(\mathcal{Z}) \subseteq \{\text{Weierstrass divisor}\} \cup \{\text{Hyperelliptic divisor}\}$.
- $\Delta_{2,j} \nsubseteq \mathcal{Z}$, for all j.

Then $[\mathcal{Z}] = 0 \in CH^1(\overline{\mathcal{M}}_g)$, hence $\mathcal{Z}$ is empty.

Moral: One has to prove the Strong Maximal Rank Conjecture not only generically over $\overline{\mathcal{M}}_g$, but also over all divisors on $\overline{\mathcal{M}}_g$.

The proof of the Strong Maximal Conjecture

Let Γ be the following tropical curve of genus g (chain of loops):



- Choose $R \subseteq K$ a DVR and $X \to K$ a smooth curve of genus g, with a regular model $\mathcal{X} \to R$, such that Γ is the metric graph associated to this degeneration (Locally, $\mathcal{X} \to \mathrm{Spec}(R)$ is given by $xy = t^{m_i}$ resp. $xy = t^{\ell_i}$ in a neighborhood of the node corresponding to v_i resp. w_i).
- Each point of Γ corresponds to a valuation of $K(X)$. This induces a map

$$K(X)^* \ni f \mapsto \mathrm{trop}(f) \colon \Gamma \to \mathbb{R},\ v \mapsto \mathrm{val}_v(f).$$

- Lelong: $\mathrm{div}(\mathrm{trop}(f)) = \mathrm{Trop}(\mathrm{div}(f))$, where $\mathrm{Trop} \colon \mathrm{Div}(X) \to \mathrm{Div}(\Gamma)$.

Definition

If $D_X \in \mathrm{Div}(X)$ and $V^{r+1} \subseteq H^0(X, D_X)$ is a linear system, its tropicalization is $\mathrm{trop}(V) := \big\{\phi \in PL(\Gamma)$ with $\mathrm{div}(\phi) + \mathrm{Trop}(D_X) \geq 0\big\}$.

- Fact: $\mathrm{trop}\left(W^r_d(X)\right) \subseteq W^r_d(\Gamma)$.

Theorem

(FJP) Let X/K be a smooth curve of genus $g = 22, 23$ whose minimal skeleton of X^{an} is Γ. Assume the lengths of Γ are general enough. Then $\phi_L \colon \mathrm{Sym}^2 H^0(X, L) \to H^0(X, L^2)$ is injective for all $L \in W^6_{g+3}(X)$.

Ingredients in the proof:

- There is a complete combinatorial description of $W^r_d(\Gamma)$. For each divisor $D' \in \mathrm{Div}(\Gamma)$ with $r(D') \geq r$, there exists a unique break divisor $D \sim D'$ with $D = (d-g)\cdot v_1 + x_1 + \cdots + x_g$, with $x_i \in \mathrm{Loop}_i$.

$$W^r_d(\Gamma) = \bigcup \big\{\mathbb{T}(\lambda) : \lambda \colon [r+1] \times [g-d+r] \to [g] \text{ standard tableau}\big\},$$

where $\mathbb{T}(\lambda)$ is a ρ-dimensional torus. The free coefficients of x_i are those corresponding to $i \notin \mathrm{Im}(\lambda)$. In particular $\dim W^r_d(\Gamma) = \rho(g, r, d)$.

• The general point of $\mathbb{T}(\lambda)$ corresponds to a vertex avoiding divisor D, i.e. for each i there exists a unique $D_i \sim D$ with $D_i \geq i \cdot v_1 + (r-i) \cdot w_g$.

• Suppose $L \in W^6_{g+3}(X)$ is such that $\phi_L\colon \mathrm{Sym}^2 H^0(X,L) \to H^0(X,L^2)$ is not injective. Choose a basis $f_0, \ldots, f_6$ of $H^0(X,L) = H^0(X,D_X)$, such that $\sum_{i,j} a_{ij} f_i f_j = 0$, for $a_{ij} \in K$. Then for any $v \in \Gamma$

$$\min_{i,j}\big\{v(a_{ij}) + \mathrm{trop}(f_i)(v) + \mathrm{trop}(f_j)(v)\big\}$$

is attained at least twice.

Definition

The piecewise linear functions $\psi_1, \ldots, \psi_n$ on Γ are tropically dependent if there exist $a_1, \ldots, a_n \in \mathbb{R}$ such that $\min_i\{\psi_i + a_i\}$ is attained at least twice for each $v \in \Gamma$.

• Set $\varphi_i := \mathrm{trop}(f_i)$. To prove Strong MRC it suffices to construct a tropical independence between the 28 (explicitly known!) PL functions

$$\big\{\varphi_i + \varphi_j)\big\}_{i \leq j}.$$

This is relatively easy in the vertex avoiding case and much harder in the remaining cases...

• Recall $\varphi_i = \text{trop}(f_i) \in PL(\Gamma)$ for $i = 0, \ldots, 6$. In the vertex avoiding case, we can choose the f_i's such that the φ_i's are explicit functions. Precisely, the slope $s_k(\varphi_i)$ along the k-th bridge of Γ is given by

$$s_k(\varphi_i) = i - (g - d + r) + \Big|\{\text{entries } \le k \text{ in column } r + 1 - i\}\Big|.$$

• One has to extend this to all divisors on Γ of degree $g + 3$ and dimension 6. For $g = 23$ the number of combinatorial types in $W^6_{26}(\Gamma)$ equals

$$\frac{23!}{9! \cdot 8! \cdot 7!} = 350,574,610.$$

When D_X is no vertex avoiding case, we no longer have the distinguished function φ_i. But for small ρ, it suffices to understand the tropicalization of certain subpencils in $H^0(X, D_X)$. The possibilities for the tropicalization of $H^0(X, D_X)$ are divided into cases, according to the combinatorial properties of these pencils. We then construct a tropical independence case-by-case, using a generalization of the algorithm that works for vertex avoiding divisors.

• Construct a tropical independence in the vertex avoiding case for $g = 22$. Take a $(3,7)$ random tableau (entry 18 is missing):

01	03	06	09	10	13	15
02	05	07	12	16	19	20
04	08	11	14	17	21	22

Depict a tropical independence $\theta = \min_{ij}\{\varphi_i + \varphi_j + c_{ij}\}$. The dots indicate the support of $D' = 2D + \mathrm{div}(\theta)$. Each of the 28 functions $\varphi_i + \varphi_j$ achieves the minimum uniquely on the component of the complement of $\mathrm{Supp}(D')$ labeled ij.