Braid varieties

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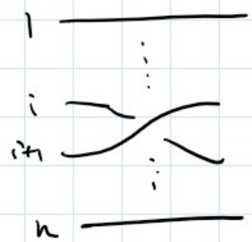
Algebraic Weaves and Braid Varieties arXiv:2012.06931

Positroid Links and Braid varieties arXiv:2105.13948

① **Braid group**



gens: $\sigma_1 \dots \sigma_{n-1}$

rels: $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

$\sigma_j\sigma_i = \sigma_i\sigma_j,\ |i-j|>1$

$\beta = \sigma_{i_1}\dots\sigma_{i_r}$ = positive braid (no σ_i^{-1})

$B_\beta(z_1\dots z_r) = B_{i_1}(z_1)\cdot\ldots\cdot B_{i_r}(z_r)$

where
$$B_i(z) = \begin{array}{c}\begin{pmatrix}1 & & & & & \\ & \ddots & & & 0 & \\ & & 0 & 1 & & \\ & & 1 & z & & \\ & 0 & & & \ddots & \\ & & & & & 1\end{pmatrix}\\ \qquad\quad i+1\end{array}$$
$n \times n$ matrix

Fact: $B_i(z_1)B_{i+1}(z_2)B_i(z_3) = B_{i+1}(z_3)B_i(z_2 - z_1z_3)B_{i+1}(z_1)$

$\Rightarrow B_\beta(z_1\dots z_r)$ is well defined up to a change of variables

Def $X(\beta) = \{z_1\dots z_r \in \mathbb{C}^r \mid B_\beta(z_1\dots z_r)\cdot w_0$ is upper-triangular$\}$

Braid variety

$$w_0 = \begin{pmatrix}0 & & 1\\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0\end{pmatrix}$$

similar spaces considered by

Broué - Michel, Deligne

Shende - Treumann - Zaslow

Mellit

Ex $\beta = \sigma_1^3$

$$\begin{pmatrix}0 & 1\\ 1 & z_1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & z_2\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & z_3\end{pmatrix}\underbrace{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}}_{w_0} =$$

$$\begin{pmatrix}0&1\\1&z_1\end{pmatrix}\begin{pmatrix}0&1\\1&z_2\end{pmatrix}\begin{pmatrix}0&1\\1&z_3\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} =$$

(Mellit)

$$\begin{pmatrix}1&z_2\\z_1&1+z_1z_2\end{pmatrix}\begin{pmatrix}1&0\\z_3&1\end{pmatrix} = \begin{pmatrix}*&*\\z_1+z_3+z_1z_2z_3&*\end{pmatrix}$$

$$X(\beta) = \{z_1+z_3+z_1z_2z_3 = 0\} \subset \mathbb{C}^3$$

$$z_1+z_3(1+z_1z_2)$$

$1+z_1z_2=0$
$z_1=0$, contradiction

$1+z_1z_2\neq 0$

$$z_3 = -\frac{z_1}{1+z_1z_2}$$

torus action
$z_1 \to t z_1$
$z_2 \to t^{-1} z_2$
$z_3 \to t z_3$

$$X(\beta) = \{1+z_1z_2 \neq 0\} \subset \mathbb{C}^2$$
also known as A_1 cluster variety.

Thm (a) $X(\beta)$ is not empty $\iff$ β contains w_0 as a subword.

contains positive braid lift of w_0

In this case, $X(\beta)$ is smooth, $\dim = \ell(\beta) - \binom{n}{2}$

(b) There is an interesting action of $(\mathbb{C}^\times)^{n-1}$ on $X(\beta)$. For a certain subtorus $T \subset (\mathbb{C}^*)^{n-1}$, the action of T is free and $X(\beta)/T$ is holomorphic symplectic.

(c) $X(\beta)$ has a smooth compactification [which depends on a choice of a braid word for β] ($\simeq$ Brick manifolds, L. Escobar)

strata = subwords of β containing w_0

Ex $\{1+z_1z_2\neq 0\} \subset \mathbb{C}^2$ compactifies to $\mathbb{C}P^1\times\mathbb{C}P^1$

subword complex (Knutson-Miller)

complement $= \{$hyperbola$\} \cup \mathbb{C}P^1 \cup \mathbb{C}P^1$
is $\mathbb{C}P^1$

Thm $w, u \in S_n$, $w \geq u$ in Bruhat order

$\cancel{\beta(w)}$, $\beta(u^{-1}w_0)$ = positive braid lifts

Then $X(\beta(w)\cdot\beta(u^{-1}w_0))$ = open Richardson variety for w, u

$\Rightarrow$ lots of interesting examples

Rmk (Knutson-Lam-Speyer) Under certain additional assumptions on w, this is isomorphic to an open positroid variety in $Gr(k,n)$.

Thm $\beta = \dots\dots \overset{z_1}{\sigma_i}\overset{z_2}{\sigma_i} \dots\dots$

$\beta' = \dots\dots \sigma_i \dots\dots \qquad \dots 1 \dots\dots = \beta''$

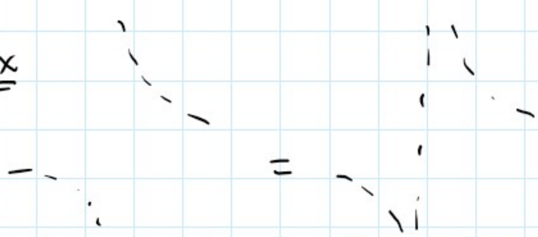
Then $X(\beta) = \underset{\{z_1 \neq 0\}}{X(\beta') \times \mathbb{C}^*} \sqcup \underset{\{z_1 = 0\}}{X(\beta'') \times \mathbb{C}}$

Cor Lots of interesting stratifications of $X(\beta)$

strata $= (\mathbb{C}^*)^{\cdots} \times \mathbb{C}^{\cdots}$

cluster transformation

Ex



$= \qquad \sqcup$

$\{1+z_1z_2 \neq 0\} \qquad \{z_1 \neq 0, 1+z_1z_2 \neq 0\} \qquad \{z_1 = 0\}$

$\mathbb{C}^* \times \mathbb{C}^* \qquad \mathbb{C}$

$(q-1)^2 + q = q^2 - q + 1$

Another stratification

$\mathbb{C}^{*2}_{\{z_2 \neq 0, 1+z_1z_2 \neq 0\}}$

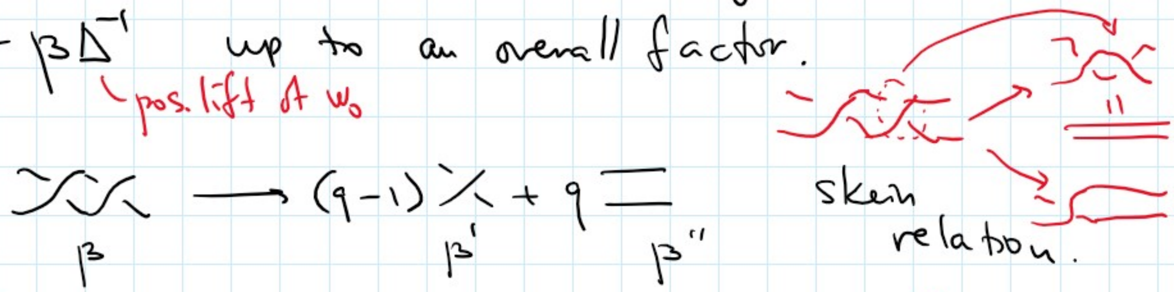
$\sqcup$

$\{z_2 = 0\}$

$\mathbb{C}^* \times \mathbb{C}^*$ $\quad (q-1)^2 \quad + \quad \mathbb{C} \quad q \quad = q^2 - q + 1$ $\quad \{t_2 = 0\}$

Cor (Kálmán) The number of points in $X(\beta)$ over a finite field $\mathbb{F}_q$ equals the lowest a-degree of the HOMFLY-PT polynomial $\rightsquigarrow P(a,q)$ of $\beta\Delta^{-1}$ up to an overall factor.

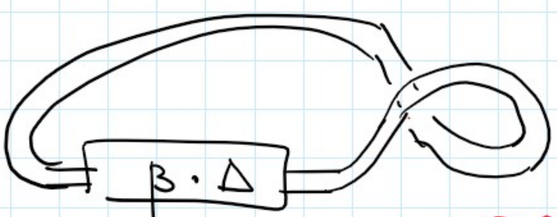
pos. lift of w_0



$\beta \longrightarrow (q-1)\,\beta' + q\,\beta''$ skein relation.

So $\#X(\beta)$ is an invariant of a link obtained by closure of $\beta\Delta^{-1}$.

Thm The variety $X(\beta)$ is an invariant of the Legendrian link:



[smooth type $= \beta \cdot \Delta \cdot \Delta^{-2} = \beta\Delta^{-1}$]

"pigtail closure"

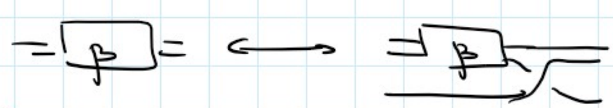
That is, if β and β' are related by braid moves, Δ-conjugation and positive stabilization then $\boxed{X(\beta) \simeq X(\beta')}$ up to $(\mathbb{C}^*)^{\cdots}$.

possibly, β and β' have different # strands

$X(\beta) \times (\mathbb{C}^*)^{N_1} \cong X(\beta') \times (\mathbb{C}^*)^{N_2}$

Δ-conjugation: $\beta \longleftrightarrow \sigma_{n-i}\,\beta\,\sigma_i^{-1}$

stabilization:



Problem: Conjugation can turn a positive braid to a non-positive which becomes positive after braid ... strand to a non-positive, which becomes positive after braid moves.

Ex Positroid varieties in $Gr(k,n)$ can be presented as $X(\beta) = X(\beta')$ where β has n strands as above and β' has k strands!

$\beta(w)\beta(u^{-1}w_0)$

Can write explicit isomorphism!

2 strand

Ex $\sigma_1^3 \longleftrightarrow$ positroid in $Gr(2,4)$

$\updownarrow$

braid on 4 strands

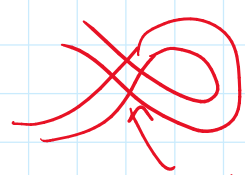
Idea of the proof of Thm:

- (Casals-Ng) Above diagram can be represented by a Legendrian link
- (Chekanov) Given a Legendrian link, one can define a differential graded algebra (dga) $\mathcal{A}$. Its homology = Legendrian link invariant
- We can define $\mathcal{A}$ when β is a non-positive braid, equivalent to positive
- Generators = crossings in a link diagram. Differential counts some disks...
- If β is a positive braid then

$$X(\beta) = \mathrm{Spec}\, \overbrace{H^0(\mathcal{A})}^{\text{commutative algebra.}}$$

Generators: $z_1 \ldots z_r \longleftarrow$ degree 0

$y_{ij} \longleftarrow$ degree 1



$\partial(y_{ij}) \approx$ lower-triangular part of

$\partial(y_{ij}) \approx$ lower-triangular part of $\quad \beta^{-1})$

$B_\beta(z_1,\dots,z_r)$ $\longleftarrow$ agrees with disks in dga $\Big]$ Kálmán

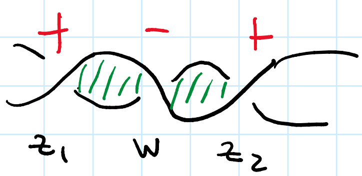
$\partial(z_i) = 0$

- If β is a non-positive braid, positive crossings.

  we get $\quad z_1 \dots z_r \longleftarrow$ degree 0 $\quad\Big\}$ as above

  $\qquad\qquad y_{ij} \longleftarrow$ degree 1

neg. crossings $\Rightarrow w_i \longleftarrow$ degree (-1).



__Ex:__ $\partial(z_1) = w + \dots$

$\partial(z_2) = -w + \dots$

$\partial(w) = 0$

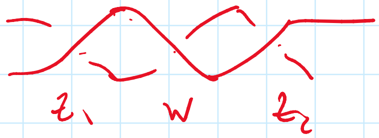
Extend by Leibniz rule: $\partial(\Phi(z_1,z_2)) = \left(\frac{\partial\Phi}{\partial z_1} - \frac{\partial\Phi}{\partial z_2}\right) w$

$\partial \rightsquigarrow$ vector field $\frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2}$

__To sum up:__

- positive crossings $+ y_{ij} \Rightarrow$ algebraic variety $X(\beta)$
- negative crossings $\Rightarrow$ commuting vector fields V_η on $X(\beta)$

__Thm__ Assume β is equivalent to a positive braid. Then the vector fields V_η integrate to a __free__ algebraic action of $\mathbb{C}^w$ on $X(\beta)$ and $\boxed{\operatorname{Spec} H^0(\mathcal{A}) = X(\beta)/\mathbb{C}^w.}$

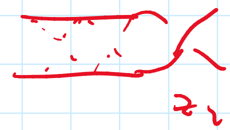


$\frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2}$ = vector field

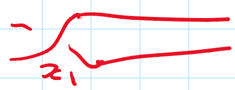
$(z_1, z_2) \longrightarrow (z_1 + t, z_2 - t)$

action of $\mathbb{C}$

Quotient: $z_1 = 0$



or $z_2 = 0$



Conclusion: $X(\beta)/\mathbb{C}^{*}$ is invariant under braid moves

conjugation
stabilization.

Galashin-Lam:

$gr^W H^*$(open positroid strata in $Gr(k,n)$)

$\quad\quad ||$

Khovanov-Rozansky homology of torus links (k,n)

Computed by Hogancamp, Mellit

$\quad\quad ||$

q,t - Catalan numbers.