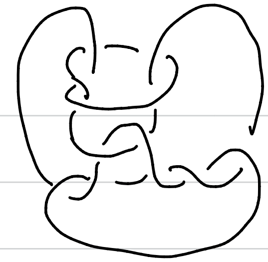
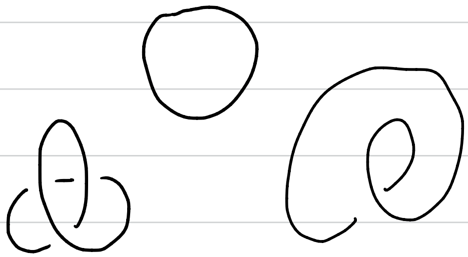
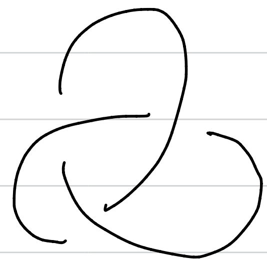
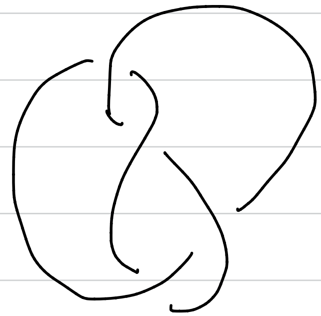
Unknotting number and satellites

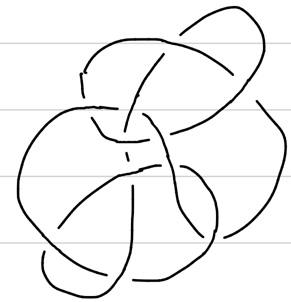
Jt w/ T. Lidman and J. Park











	unknot	trefoil	figure eight	Conway knot
$c(K)$	0	3	4	11
$u(K)$	0	1	1	1
$\Delta_K(t)$	1	$t-1+t^{-1}$	$t-3+t^{-1}$	1
$\sigma(K)$	0	-2	0	0

<u>crossing number</u> $c(K)$ = min # of crossings in any diagram for K

<u>unknotting number</u> $u(K)$ = min # of crossing changes needed to unknot K

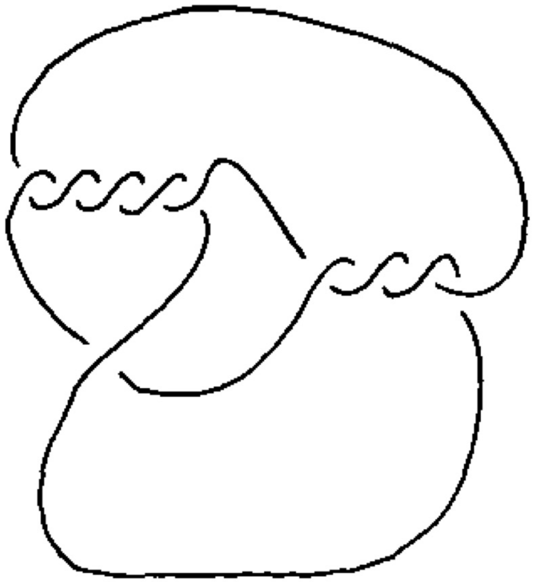
<u>Alexander polynomial</u> $\Delta_K(t)$ defined in terms of homology of infinite cyclic cover $\tilde{X}_K$

<u>signature</u> $\sigma(K)$ signature of a symmetric bilinear form on $H_1(\tilde{X}_K)$

<u>Thm</u> $\left|\frac{\sigma(K)}{2}\right| \leq u(K)$

How hard is it to determine unknotting number?

Bleiler '84



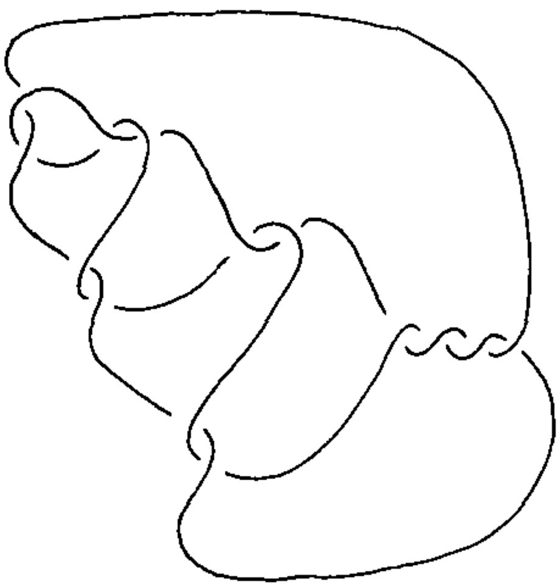
K

10 crossings

Exercise Check that no two crossing changes in this diagram result in unknot

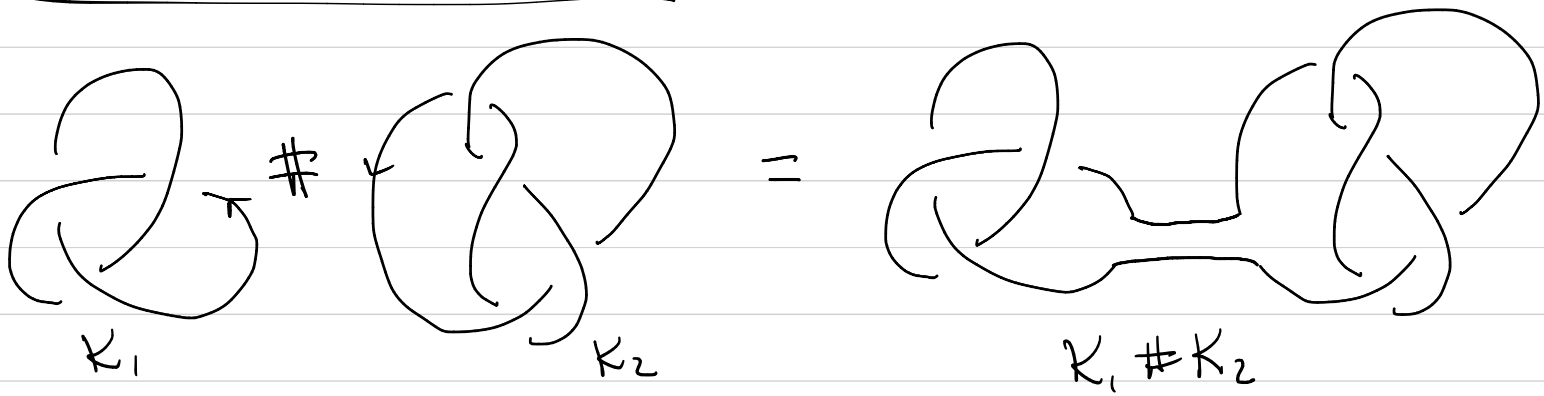
different diagram for same knot

14 crossings



Exercise Find 2 crossings to change to unknot K

Operations on knots



connected sum

crossing number

$c(K_1 \# K_2) \leq c(K_1) + c(K_2)$

$\geq$ open

Lackenby '08

$$c(K_1 \# K_2) \geq \frac{c(K_1) + c(K_2)}{152}$$

unknotting number

$u(K_1 \# K_2) \leq u(K_1) + u(K_2)$

$\geq$ open

Scharlemann '85

$u(K_1 \# K_2) \geq 2$

K_1, K_2 non-trivial

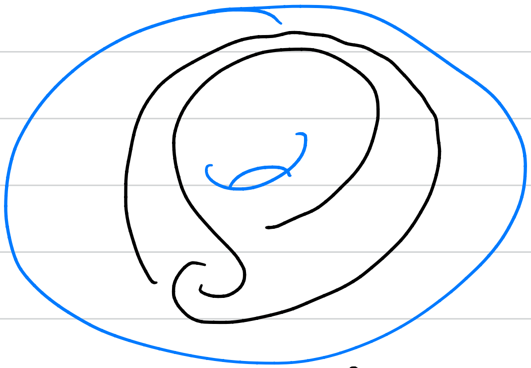
Alexander polynomial

$$\Delta_{K_1 \# K_2}(t) \doteq \Delta_{K_1}(t) \cdot \Delta_{K_2}(t)$$

signature

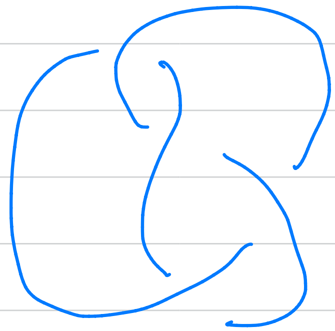
$$\sigma(K_1 \# K_2) = \sigma(K_1) + \sigma(K_2)$$

More operations on knots: satellites



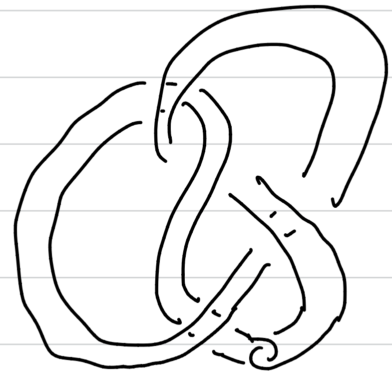
$P \subset S^1 \times D^2$

pattern

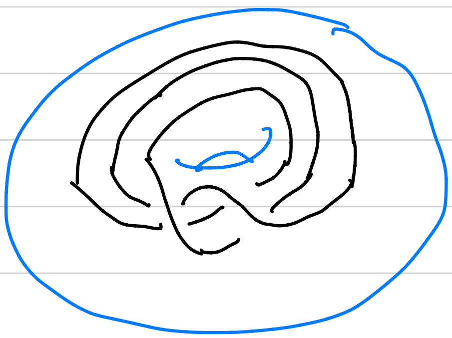


companion K

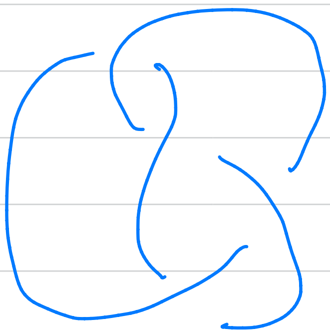
$\Rrightarrow$



$P(K)$

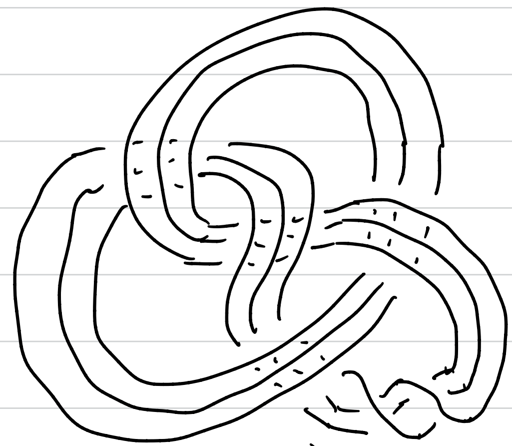


P

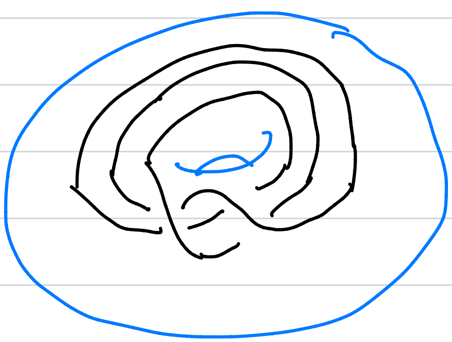


K

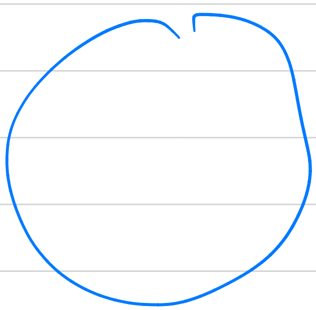
$\Rrightarrow$



$P(K)$

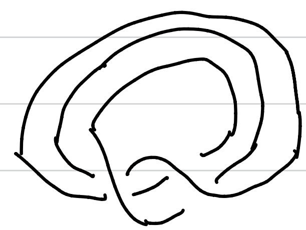


P

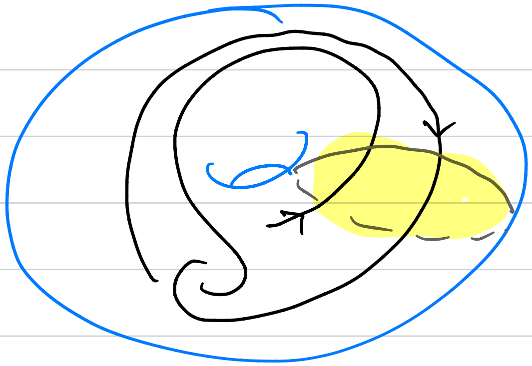


K

$\Rrightarrow$



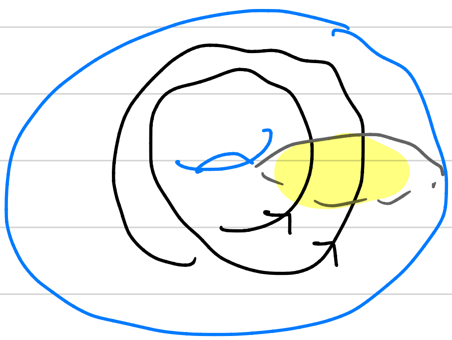
$P(K)$



Whitehead double

$w_g(P) = 2$

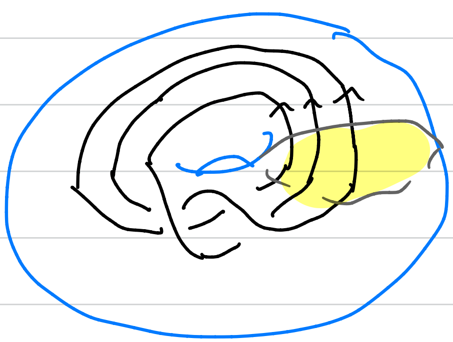
$w_a(P) = 0$



(2,1)-cable

$w_g(P) = 2$

$w_a(P) = 2$



(3,2)-cable

$w_g(P) = 3$

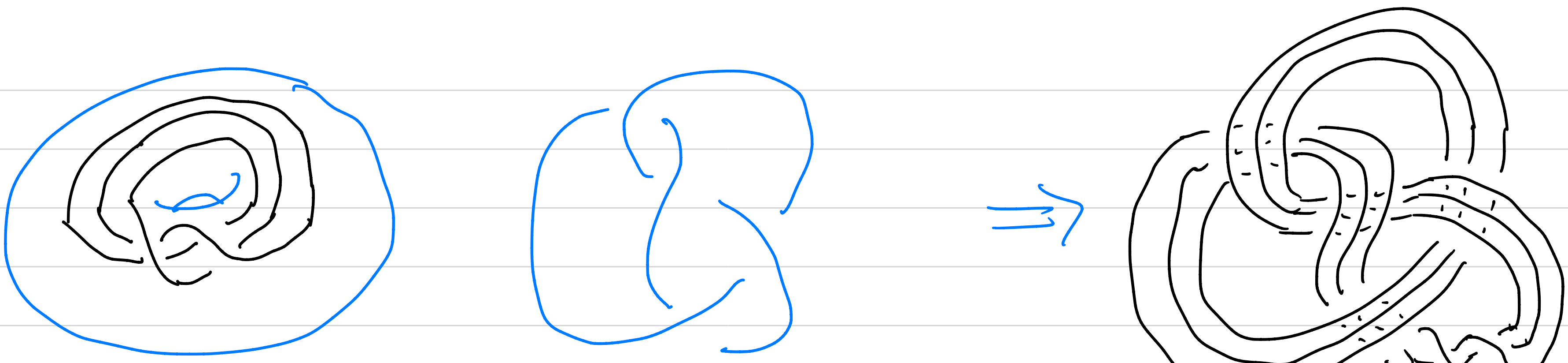
$w_a(P) = 3$

<u>geometric winding # of P</u>

$w_g(P)$ = minimum # of intersection pts between P and a meridional disk

<u>algebraic winding # of P</u>

$w_a(P)$ = minimum # of intersection pts between P and a meridional disk, counted w/ sign



$P \subset S^1 \times D^2$ $\qquad$ K $\qquad$ $P(K)$

$$\Delta_{P(K)}(t) = \Delta_K(t^{w}) \cdot \Delta_{P(U)}(t)$$

where $w = w_a(P)$ and U = unknot

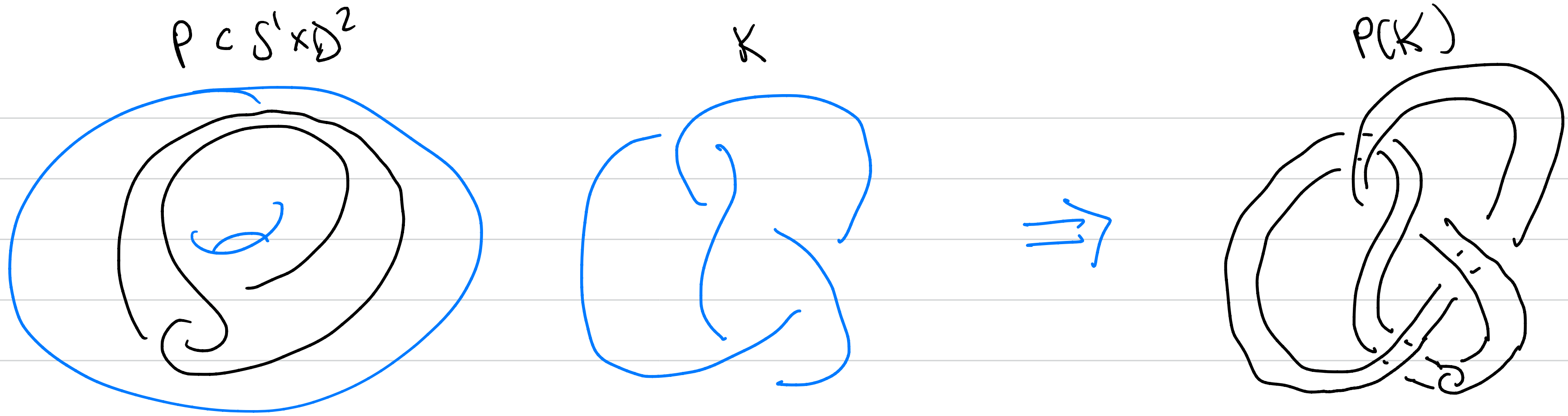
similar formula for σ

Open Q: crossing number $c(P(K)) \geq c(K)$?

$c(P(K)) \geq w_g^2\, c(K)$?

$c(P(K)) \geq c(P(U))$?

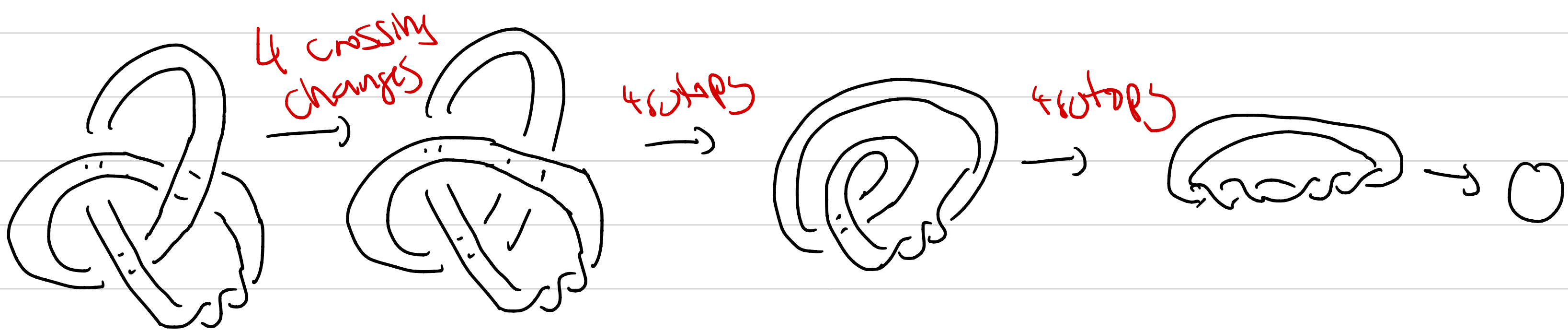
$$c(P(K)) \geq \frac{c(K)}{10^{13}}$$ Lackenby '14

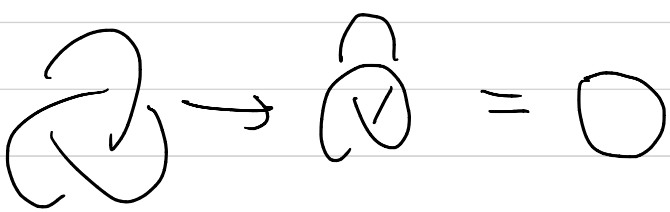


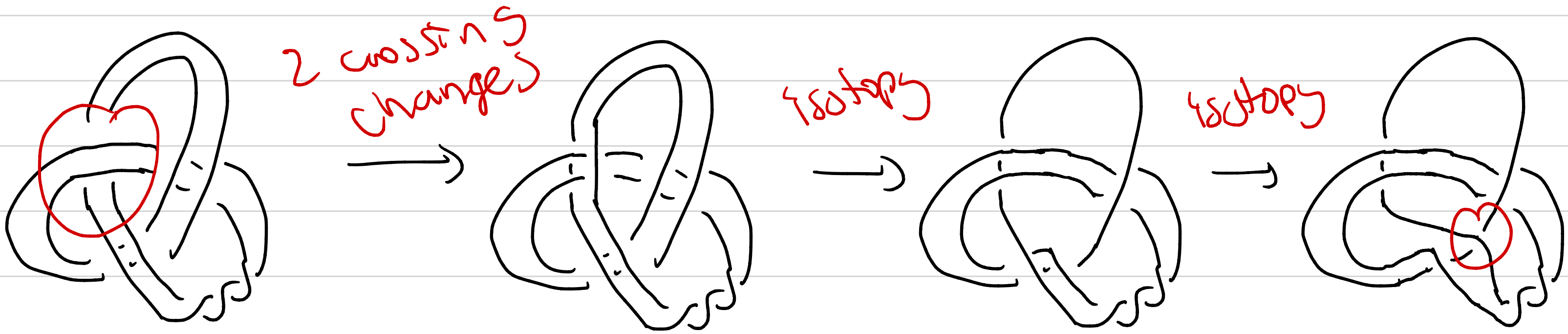
Note: For P = Whitehead double and $K \neq$ unknot, unknotting # $u(P(K)) = 1$

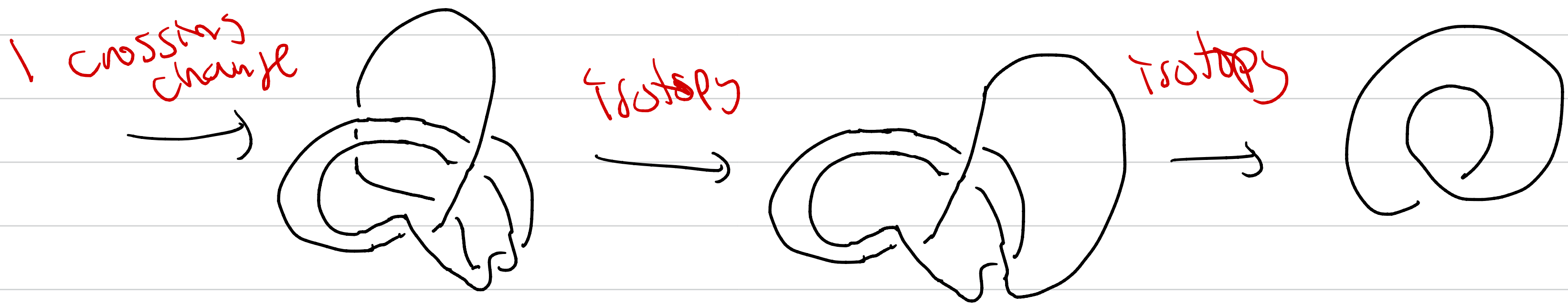
If $w_a(P) \neq 0$, then $u(P(K)) \geq 2$ Scharlemann–Thompson '89

Naive guess: $u(P(K)) \geq w_a^2\, u(K)$









$\Rightarrow$ naive guess was wrong

Open Q: If $K \neq$ unknot, $u(P(K)) \geq w_a(P)$?

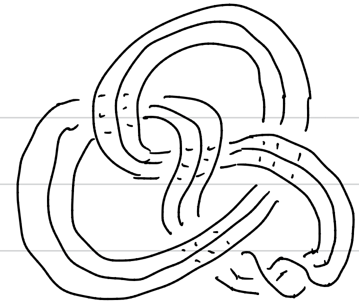
<u>Thm</u> (H-Lidman-Park)

Let $K_{m,n}$ denote the (m,n)-cable of K (winding #)

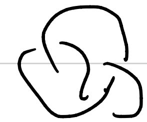
If $K \neq$ unknot, then

$u(K_{m,n}) \geq m$ $\qquad m = w_a(P)$

Proof relies on knot Floer homology
(Ozsváth–Szabó, Rasmussen)



(2,3)-cable of



$K \rightsquigarrow \widehat{HFK}(K) = \bigoplus_{i,j} \widehat{HFK}_{i,j}(K)$ bigraded vector space

$HFK^-(K)$ finitely generated graded module over $\mathbb{F}[U]$ $\quad \mathbb{F} = \mathbb{Z}/2\mathbb{Z}$

$\deg U = -2$

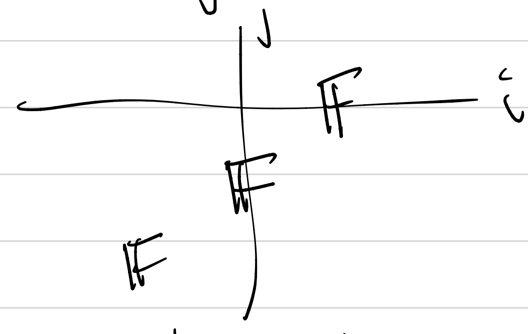
$$HFK^-(K) \cong \mathbb{F}[U] \oplus \bigoplus_i \mathbb{F}[U]/U^{n_i}$$

<u>Thm</u> (Ozsváth–Szabó, Rasmussen)
$\widehat{HFK}(K)$ categorifies $\Delta_K(t)$

$$\Delta_K(t) = \sum_{i,j} (-1)^j t^i \dim \widehat{HFK}_{i,j}(K)$$

<u>Ex</u> K = trefoil

$\Delta_K(t) = t - 1 + t^{-1}$



$t^{-1} - 1 + t$

K = fibered?

$$HFK^-(K) \cong \mathbb{F}[U] \oplus \mathbb{F}[U]/U$$

$$HFK^-(K) \cong \mathbb{F}[U] \oplus \bigoplus_{i=1}^{M} \mathbb{F}[U]/U^{n_i}$$

Define $\operatorname{Ord}(K) = \cancel{\min}\ \max \{n_i\}$

$HFK^-(\text{unknot}) = \mathbb{F}[U]$

$\operatorname{ord}(\text{unknot}) = 0$

<u>Thm</u> (OS) Knot Floer homology detects the unknot

$$\operatorname{Ord}(K) = 0 \iff K = \text{unknot}$$

<u>Thm</u> (Alishahi - Eftekhary)

$$u(K) \geq \operatorname{Ord}(K)$$

Recall: $\Delta_{P(K)}(t) = \underbrace{\Delta_K(t^w)}_{\text{stretches } \Delta_K \text{ by a factor of } w} \cdot \Delta_{P(U)}(t)$ $\qquad w = w_a(P)$

Thm (Hanselman-Rasmussen-Watson)
The knot Floer complex can be interpreted as an immersed Lagrangian in $T^2 - \{pt\}$ and (m,n)-cabling has the effect of stretching m copies of this curve by a factor of m and shifting by n

Our theorem relies on interpreting the effects of Hanselman-Rasmussen-Watson on $\mathrm{Ord}(K)$

Main idea:

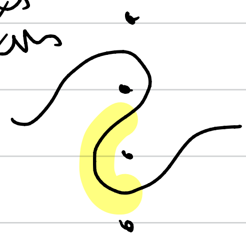
$K \neq$ unknot $\Rightarrow \mathrm{Ord}(K) \geq 1 \Rightarrow \mathrm{Ord}(K_{m,n}) \geq m$

(involves interpreting $\mathrm{Ord}(K)$ as an immersed Lagrangian and doing case analysis)

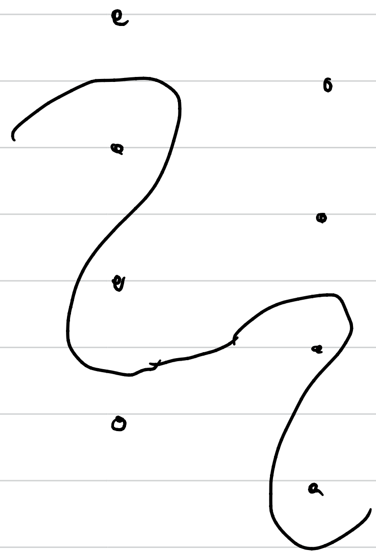
Ex K = trefoil

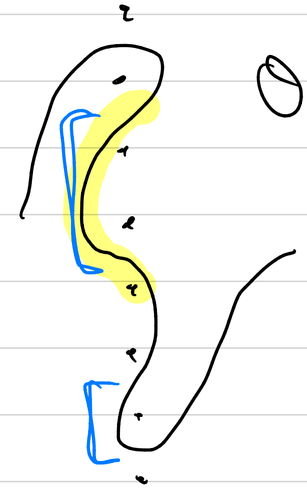
(2,3)-cable

immersed Lagrangian for trefoil



$\mathrm{Ord}(K) = 1$





$\mathrm{Ord}(K_{2,3}) \geq 2$

$\mathbb{F}[u]/u^2$

$\mathbb{F}[u]/u$

4-ish cases

$\mathrm{Ord}(K) = 2$