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# KHOVANOV HOMOLOGY, KHOVANOV HOMOTOPY

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[1]RL was supported by NSF Grant DMS-1810893
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# PLAN FOR THE TALK

1. Khovanov homology—what it is, how it's built.
2. Building stable homotopy refinements of chain complexes in general, and the Khovanov complex specifically.
3. Computations of the stable homotopy refinement of Khovanov homology.
4. Functoriality of Khovanov homology and its stable homotopy refinement.

Most of this talk will be hand-written.

# STRUCTURE OF KHOVANOV HOMOLOGY

- Knot diagram $K \rightsquigarrow$ abelian groups $Kh^{i,j}(K)$, $i,j \in \mathbb{Z}$ (finitely many non-zero). [Khovanov]
  $K \sim K' \quad \rightsquigarrow Kh^{i,j}(K) \cong Kh^{i,j}(K')$
- Jones polynomial $V_K(q) = \sum (-1)^i q^j \operatorname{rank}(Kh^{i,j}(K))$
- Cobordism $\Sigma : K \to K' \rightsquigarrow F_\Sigma : Kh^{i,j}(K) \to Kh^{i,j-\chi(\Sigma)}(K')$ [Jacobsson, Khovanov, Bar-Natan, ...]
  $(\Sigma \subset [0,1] \times \mathbb{R}^3,\ \partial\Sigma = (0 \times K) \cup (1 \times K'))$. $\quad \Sigma \sim \Sigma' \Rightarrow F_\Sigma = \pm F_{\Sigma'}$.

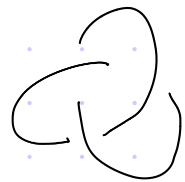
## Example Application

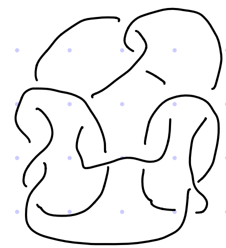
**Def.** (Gordon '81) $\Sigma : K \to K'$ is a ribbon concordance if $\Sigma \cong [0,1] \times S^1$ and $\Sigma \to [0,1] \times \mathbb{R}^3 \to [0,1]$ Morse w/ no deaths

**Conj.** (Gordon '81) Ribbon concordance is a strict partial order

**Thm.** (Levine-Zemke '19) Then $F_\Sigma$ is a split injection.

**Cor.** If $K$ is alternating then crossing # of $K \leq$ crossing # of $K'$





(KT)

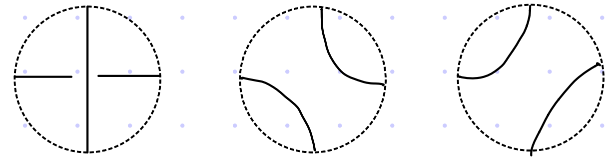
| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 9 | | | | $\mathbb{Z}$ |
| 7 | | | $\mathbb{F}_2$ | |
| 5 | | | $\mathbb{Z}$ | |
| 3 | $\mathbb{Z}$ | | | |
| 1 | $\mathbb{Z}$ | | | |

| | -6 | -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 9 | | | | | | | | | | | | $\mathbb{Z}$ |
| 7 | | | | | | | | | | | $\mathbb{Z}$ | $\mathbb{F}_2$ |
| 5 | | | | | | | | | | $\mathbb{Z}$ | $\mathbb{Z}\oplus\mathbb{F}_2$ | |
| 3 | | | | | | | | $\mathbb{Z}$ | $\mathbb{Z}^2$ | $\mathbb{Z}\oplus\mathbb{F}_2$ | | |
| 1 | | | | | | | $\mathbb{Z}^2$ | $\mathbb{Z}\oplus\mathbb{F}_2$ | $\mathbb{Z}\oplus\mathbb{F}_2^2$ | | | |
| -1 | | | | | | $\mathbb{Z}$ | $\mathbb{Z}^3\oplus\mathbb{F}_2$ | $\mathbb{Z}^2\oplus\mathbb{F}_2$ | | | | |
| -3 | | | | | $\mathbb{Z}^2$ | $\mathbb{Z}^2\oplus\mathbb{F}_2$ | $\mathbb{Z}\oplus\mathbb{F}_2$ | | | | | |
| -5 | | | | $\mathbb{Z}$ | $\mathbb{Z}\oplus\mathbb{F}_2^2$ | $\mathbb{Z}\oplus\mathbb{F}_2$ | | | | | | |
| -7 | | | $\mathbb{Z}$ | $\mathbb{Z}^2\oplus\mathbb{F}_2$ | $\mathbb{Z}$ | | | | | | | |
| -9 | | $\mathbb{Z}$ | $\mathbb{Z}\oplus\mathbb{F}_2$ | | | | | | | | | |
| -1 | | $\mathbb{Z}\oplus\mathbb{F}_2$ | | | | | | | | | | |
| 1 | $\mathbb{Z}$ | | | | | | | | | | | |

# CONSTRUCTION OF KHOVANOV HOMOLOGY

$V = \mathbb{Z}[X]/(X^2)$. $\quad m: V \otimes V \to V$.

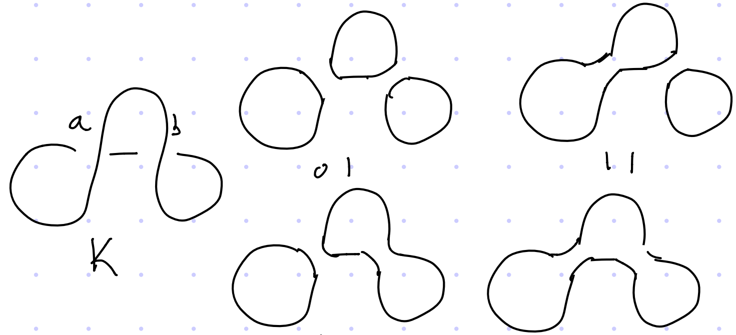
$\Delta: V \to V \otimes V \quad \Delta(1) = 1 \otimes X + X \otimes 1 \quad \Delta(X) = X \otimes X$




Knot diagram $K \rightsquigarrow$ cube of resolutions

Apply the TQFT $V$, i.e.:

$S^1 \rightsquigarrow V \quad \sqcup \rightsquigarrow \otimes \quad$ merge $\rightsquigarrow m \quad$ split $\rightsquigarrow \Delta$




<u>Exercise</u>: This gives a commutative cube.

Make it anticommute by adding signs to some edges, take the total complex, and take homology.

$$\begin{tikzcd}
V \otimes V \otimes V \arrow[r, "\mathrm{Id} \otimes m"] & V \otimes V \\
V \otimes V \arrow[u, "\Delta \otimes \mathrm{Id}"] \arrow[r, "\Delta"'] & V \arrow[u, "-m"']
\end{tikzcd}$$

# STABLE HOMOTOPY REFINEMENTS...

<u>Idea.</u> (Cohen–Jones–Segal, Blanc–Johnson–Turner, ...) Instead of a complex of abelian groups, build a complex of (stable) spaces.

$$\begin{tikzcd} 0 & \mathbb{Z} \arrow[l] & \mathbb{Z} \arrow[l, "\cdot 6"'] & 0 \arrow[l] \arrow[r, dashed] & pt & S^N \arrow[l] & S^N \arrow[l, "f"'] & pt \arrow[l] \end{tikzcd}$$

$\deg(f) = 6.$

$H_*$: $0 \quad \mathbb{Z}/6 \quad 0 \quad 0$

$C_*^{cell}$, Cone: $S^N \cup_f D^{N+1}$, $\tilde{H}_*$

e.g. $0 \leftarrow \mathbb{Z} \xleftarrow{[3\ \ -6]} \mathbb{Z}\oplus\mathbb{Z} \xleftarrow{\left[\begin{smallmatrix}4\\2\end{smallmatrix}\right]} \mathbb{Z} \leftarrow 0$

$$\begin{tikzcd} S^N & S^N \vee S^N \arrow[l, "f"'] & S^N \arrow[l, "g"'] \arrow[ll, bend right, "\text{Null htpy of } g\circ f"'] \end{tikzcd}$$

Iterated cone

$X = \mathrm{Cone}(\underbrace{F}_{?} : \underbrace{\mathrm{Cone}(g)}_{(S^N \vee S^N)\cup_g D^{n+1}} \to S^N)$

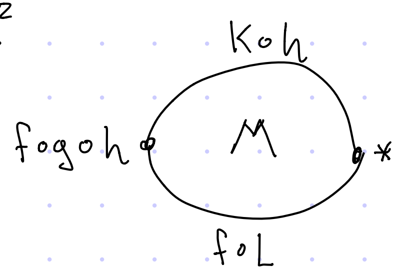
X depends on the null homotopy.

e.g. $\mathbb{Z} \leftarrow 0 \leftarrow \mathbb{Z} \rightsquigarrow S^N \leftarrow pt \leftarrow S^N.$

Null htpy is a map $\Sigma S^N \to S^N$.

Constant $\rightsquigarrow S^N \vee S^{N+2}$ Hopf $\rightsquigarrow \Sigma^{N-2}\mathbb{C}P^2.$

4-step: $$\begin{tikzcd} \vee S^N & \vee S^N \arrow[l, "f"'] & \vee S^N \arrow[l, "g"'] \arrow[ll, bend left, "f\circ g \sim_K *"] & \vee S^N \arrow[l, "h"'] \arrow[ll, bend left, "g\circ h \sim_L *"] \arrow[lll, bend left=60, "M"] \end{tikzcd}$$



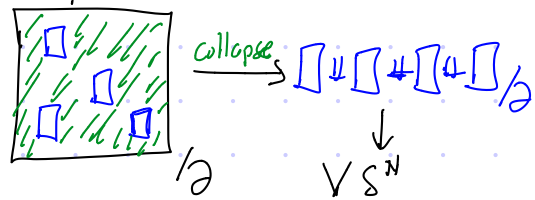

etc.

# …OF KHOVANOV HOMOLOGY

So, the problem of refining chain complexes has $\pi_*^s$ built into it (bad? good?).

Recall that the Khovanov complex is positive.

A <u>box map</u> is a map

$S^N =$ 


$$\begin{tikzcd} V\otimes V\otimes V \arrow[d, "Id\otimes m"'] & V\otimes V \arrow[l, "\Delta\otimes Id"'] \arrow[d, "-m"] \\ V\otimes V & V \arrow[l, "\Delta"] \end{tikzcd}$$

The space of box maps with given combinatorics is contractible. So refining a complex where all maps are box maps only requires a new choice of first homotopies (3-step) and only obstruction is for second homotopies (4-step)

<u>Thm.</u> (L-Sarkar, Hu-Kriz-Kriz, Lawson-L-Sarkar) For the Khovanov complex, such a refinement exists, and the stable homotopy type of the iterated mapping cone $X^j_{Kh}(K)$ is a knot invariant.

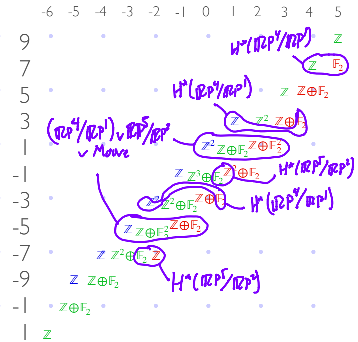
$\widetilde{H}^i(X^j_{Kh}(K)) = Kh^{i,j}(K)$. $\rightsquigarrow$ Steenrod operations.

e.g.

$$Sq^n : Kh^{i,j}(K) \longrightarrow Kh^{i+n,j}(K). \qquad Sq^1 = \beta.$$

L-Sarkar: a formula for $Sq^2$. Cantero-Marán: formula for $Sq^n$.

# COMPUTATIONS




## Boring examples:

**Exercise.** Suppose $\tilde{H}^i(X)=0$ for $i \notin \{n, n+1\}$ and $H^n(X)$ is torsion-free, and $\pi_1(X)=0$. Then $X \simeq M(H_n(X), n) \vee M(H_{n+1}(X), n+1)$.

(Moore spaces)

**Corollary.** If $Kh(K)$ is thin (e.g. $K$ alternating) then $\mathcal{X}(K) \simeq \bigvee$ of Moore spaces

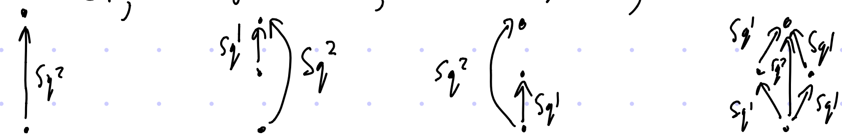
|  | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 9 |  |  |  | $\mathbb{Z}$ |
| 7 |  |  | $\mathbb{F}_2$ |  |
| 5 |  |  | $\mathbb{Z}$ |  |
| 3 | $\mathbb{Z}$ |  |  |  |
| 1 | $\mathbb{Z}$ |  |  |  |

## Less boring examples.

**Thm.** (Whitehead, Chang) If $\tilde{H}^i(X)=0$ for $i \notin \{n, n+1, n+2\}$ and $\tilde{H}^i(X)$ has no odd torsion then the stable homotopy type of $X$ is determined by $H^*(X)$, $Sq^1$, and $Sq^2$.

In fact, $X$ is a wedge sum of Moore spaces,

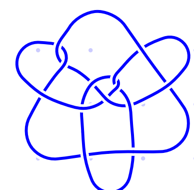
$$\Sigma^{n-2}\mathbb{C}P^2,\quad \Sigma^{n-2}(\mathbb{R}P^4/\mathbb{R}P^1),\quad \Sigma^{n-3}(\mathbb{R}P^5/\mathbb{R}P^2),\quad \Sigma^{n-1}(\mathbb{R}P^2\wedge\mathbb{R}P^2)$$




e.g. $\mathcal{X}(T_{3,4})$ has an $\mathbb{R}P^4/\mathbb{R}P^2$.

- So, computing $Sq^2$ tells you $\mathcal{X}_{Kh}(K)$ if $Kh(K)$ has width 3.

- Seed: $\exists$ knots w/ isomorphic $Kh$ (and HOMFLY-PT) but different $\mathcal{X}_{Kh}$. e.g.



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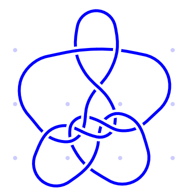

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(Diagrams from SnapPy and KLO)

# FUNCTORIALITY

<u>Thm.</u> (LLS) Given a link cobordism $\Sigma : K \to K'$ there is a well-defined up to sign homotopy class of maps
$X_{Kh}(\Sigma) : X^j_{Kh}(K) \to X^{j-\chi(\Sigma)}_{Kh}(K')$. $X_{Kh}(Id) \sim Id$ and $X_{Kh}(\Sigma' \circ \Sigma) \sim \pm X_{Kh}(\Sigma') \circ X_{Kh}(\Sigma)$.

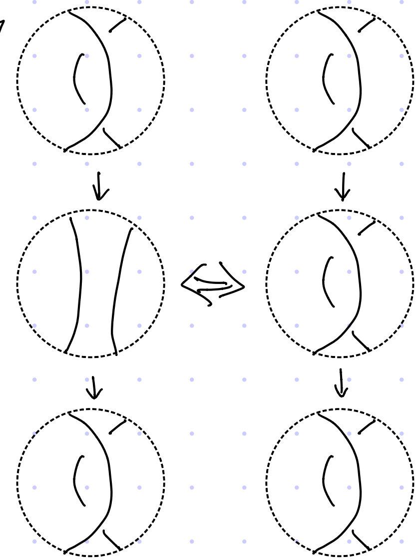
$$\begin{tikzcd} H_i X^j_{Kh}(K) \arrow[r, "X(\Sigma)"] \arrow[d, "\cong"] & H_i X^{j-\chi(\Sigma)}(K') \arrow[d, "\cong"] \\ Kh_{i,j}(K) \arrow[r, "F_{Kh}"'] & Kh_{i,j-\chi(\Sigma)}(K') \end{tikzcd}$$

<u>pf.</u> sketch

Similar to Khovanov's proof of functoriality of Kh.

Represent $\Sigma$ by a <u>movie</u>. Associate maps to births, deaths, saddles, and Reidemeister moves.

Verify movie moves give homotopic maps, by using $X_{Kh}$ of tangles. ∎



Can detect maps invisible to Kh. e.g., Hopf-like map $(S^5 \to \Sigma^2 \mathbb{R}P^2) \in \pi_5(\Sigma^2 \mathbb{R}P^2) = \mathbb{Z}/2$.

<u>Question.</u> Are there cobordisms $\Sigma, \Sigma' : K \to K'$ s.t. $F_{Kh}(\Sigma) = F_{Kh}(\Sigma')$ but $X_{Kh}(\Sigma) \not\sim X_{Kh}(\Sigma')$?

(cf. Sundberg–Swann)

<u>Question'.</u> What are useful, computable invariants of maps in this context?