Stable cohomology of complements of discriminants — with application to moduli spaces

§ Discriminants

Discriminant: locus of degenerate elements in a vector space of functions

this talk: $/\mathbb{C}$

Example: $V_d = \mathbb{C}[x_0, \dots, x_m]_d \ni f \rightsquigarrow$ hypersurface $\mathbb{V}(f) \subset \mathbb{P}^m$ if $f \neq 0$

discriminant $\Sigma_d \subset V_d$, $\Sigma_d = \left\{ \begin{matrix} \text{singular} \\ \text{polynomials} \end{matrix} \right\} \ni f$ s.th. $\frac{\partial f}{\partial x_0}(p) = \dots = \frac{\partial f}{\partial x_m}(p) = 0$ for some $p \in \mathbb{P}^m$

Complement: $GL(m+1) \circlearrowleft X_d = V_d \setminus \Sigma_d \rightsquigarrow X_d / GL(m+1)$ coarse moduli space of degree d hypersurfaces

More in general: M smooth proj. variety
L very ample line bundle

$V_{d,(M,L)} := H^0(M, L^{\otimes d})$

$\Sigma_{d,(M,L)} \subset V_{d,(M,L)}$, $\Sigma_{d,(M,L)} = \{\text{singular sections}\}$ discriminant

$X_{d,(M,L)} = V_{d,(M,L)} \setminus \Sigma_{d,(M,L)}$ up to scaling: parametrizes smooth divisors in $|L^{\otimes d}|$.

Application: moduli spaces of smooth divisors on M.

In that case: G-action on M inducing $G \circlearrowright V_{d,(M,L)}$

$\rightsquigarrow \mathcal{M}_d = X_{d,(M,L)} / G$.

Stabilization questions:

Q1. Is there an isom. $H^k(X_{d,(M,L)}) \cong H^k(X_{d',(M,L)})$ for $k \ll d < d'$?

Q2. What is $H^k(X_{d,(M,L)})$ in this stable range?

Q3. Can we use this to describe the cohom. of $\mathcal{M}_d$ in small degree k?

Betti numbers
Hodge structures

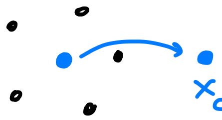
§ Background

Classical result: Arnol'd, 1970

$$\left\{ \begin{array}{l} \text{monic polynomials} \\ \text{of degree } d \\ \text{in one var.} \end{array} \right\} \cong \mathbb{C}^d \supset Y_d$$

locus of polynomials w/o multiple roots

Continuous map $Y_d \longrightarrow Y_{d+1}$ adding a point far away



$$x_{d+1} = \underbrace{\frac{x_1 + \cdots + x_d}{d}}_{x_0} + \max |x_i - x_0| + 1$$

Arnol'd this map defines an iso $H^\bullet(Y_d;\mathbb{Z}) \cong H^\bullet(Y_{d+1};\mathbb{Z})$ provided $d \geq 2\bullet - 2$.

finite group if $\bullet \geq 2$

Vakil-Wood 2015: in the Grothendieck group of varieties

$$\boxed{\frac{[X_{d,(M,L)}]}{[V_{d,(M,L)}]}}$$

stabilizes to a motivic ζ function of M in a suitable completion of a localization of the Grothendieck ring of varieties

CONSTRAINTS on the answer to Q2. (mainly on Hodge structures)

§ Results

Thm (OT, 2021) For all (M,L), $H^k(X_{d,(M,L)}, \mathbb{Z})$ stabilizes for $k < \lfloor \frac{d+1}{2} \rfloor$.

Moreover,

(**) IF the Vassiliev spectral sequence $E_1^{p,q} \Rightarrow \bar{H}^{-p-q}(X_d, \mathbb{Q})$ degenerates at E_1,

THEN $H^\bullet(X_d; \mathbb{Q})_{\text{STABLE}} \cong$ free graded-commutative algebra generated by $H^{\bullet-1}(M) \otimes \underbrace{\mathbb{Q}(-1)}_{\text{Hd structure}}$

(For mixed Hd structures, the isom. is only after passing to the assoc. graded)

Remarks:

(1) Vassiliev expected degeneration to occur even outside the stable range (stronger than (**)).

(2) Aumonier: different approach: alternative proof of stabilisation also implying (**).

(3) By a direct computation, one gets (**) in special cases:

(OT, 2014) $\quad (M,L) = (\mathbb{P}^m, \mathcal{O}(1))$

$$H^k(X_d; \mathbb{Q}) \cong H^k(GL(m+1); \mathbb{Q}) \quad k < \frac{d+1}{2}$$

In particular, $H^k\big(\underbrace{X_d / GL(m+1)}_{\text{mod. space of smooth hypersurf.}}\big) = 0$ for $0 < k < \frac{d+1}{2}$

Other easy case: C is a curve $\rightsquigarrow$ nice explicit formula for $H^k(X_{d,(C,L)}; \mathbb{Q})$ for d large.

Proof of Thm: Vassiliev's method (1999)

$H^k(X_d) \cong \bar{H}_{2 X_d - k - 1}(\Sigma_d)$ for $k > 0$

Borel-Moore homology

Vassiliev:

classification of sing. loci $\rightsquigarrow$ spectral seq. converging to $\bar{H}_\bullet(\Sigma)$

p small, finite conf. of p sing. pts

$$E^1_{p,q} = \bar{H}_{q+1}\big(\{(f,x) : \underbrace{x \subset \mathrm{Sing}(f)}_{\text{unord. conf. of } p \text{ distinct pts}}\}; \pm\mathbb{Z}\big), \quad f \in \Sigma_d$$

Lemma: $\{(f,x)\} \longrightarrow \mathrm{Sym}^p M \smallsetminus (\mathrm{diag.})$ is a vector bundle of the exp. dim. $\dim V_d - p(m+1)$ for $p \le \frac{d+1}{2}$.

To prove the Thm, one needs to detect the range in $H^\bullet(X_d)$ which is completely determined by this first block of $E^1_{p,q}$ with $p \le \frac{d+1}{2}$.

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§ Application:

$\mathcal{T}_g = \{$trigonal curves of genus $g\}$

smooth curve $C \xrightarrow[3:1]{} \mathbb{P}^1$

$\mathbb{F}_n \xrightarrow{F:\ \text{fiber}} \mathbb{P}^1$, $E \subset \mathbb{F}_n$: $E^2 = -n$

Maroni invariant of the curve

Then $C \hookrightarrow \mathbb{F}_n = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n))$ as a smooth divisor of class $3E + dF$ with $d = \frac{g+3n+2}{2}$.

Results: (A. Zheng):

- $H^k(X_{(\mathbb{F}_n, 3E+dF)}; \mathbb{Q})$ stabilizes for $k < \frac{d-3n+1}{2}$ + explicit description

- $\mathcal{T}_{g,n} = X_{\mathbb{F}_n, 3E+dF} / G$ Trigonal curves of Maroni inv. n

$G = \mathrm{Aut}(\mathbb{C}[x, y, z])$
$\phantom{G = \mathrm{Aut}(\mathbb{C}[}1 \ \ 1 \ \ n$

$H^\bullet(\mathcal{T}_{g,n})$ stabilizes for $k < \left\lfloor \frac{g-3n+2}{4} \right\rfloor$

$$H^i(\mathcal{T}_{g,n}) = \begin{cases} \mathbb{Q} & i=0 \\ \mathbb{Q}(-1) & i=2 \\ \mathbb{Q}(-3) & i=5 \\ \mathbb{Q}(-4) & i=7 \end{cases}$$ $n \geq 1$

- $\mathcal{T}_g = \bigcup_{n \equiv g(2)} \mathcal{T}_{g,n}$

$H^\bullet(\mathcal{T}_g; \mathbb{Q})$ stabilizes in deg. $k < \lfloor \frac{g}{4} \rfloor$ and

$$H^\bullet(\mathcal{T}_g; \mathbb{Q})_{\text{stable}} \cong \underbrace{A^{\bullet/2}(\mathcal{T}_g; \mathbb{Q})}_{\text{Canning-Larson 2021}} = \mathbb{Q}[x_1]/(x_1^3).$$

$\deg x_1 = 2$