Thin Links and Conway Spheres

joint work with
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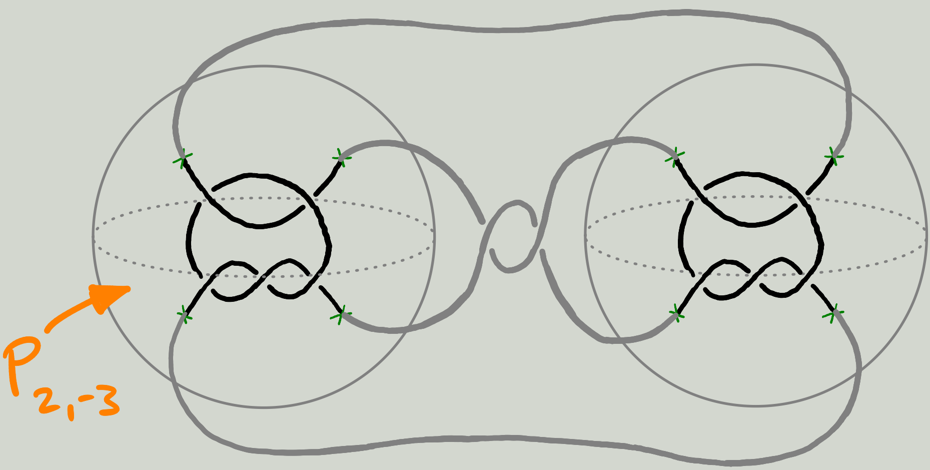
and Liam Watson

§1 A baby theorem



$$(S^2_1, 4 \text{ points}) \xrightarrow{\varphi} (S^2_1, 4 \text{ points})$$

Let $L = P_{2,-3} \cup_\varphi P_{2,-3}$

baby theorem: [Kotelskiy-Watson-Z]

$\widetilde{Kh}(L)$ is thin $\Leftrightarrow$ $\widehat{HFK}(L)$ is thin

Khovanov homology

knot Floer homology

§ 2 Thinness of $\widetilde{Kh}$ and $\widehat{HFK}$

$$\{\text{links in } S^3\} \longrightarrow \left\{\begin{array}{c}\text{finite-dim. bigraded}\\ \text{vector spaces}\end{array}\right\}$$

$$L \longmapsto \widetilde{Kh}(L) \text{ and } \widehat{HFK}(L)$$

[Khovanov] [Ozváth-Szabó, Rasmussen]

We will only consider the δ-grading:

$$\widetilde{Kh}(L) = \bigoplus_{\delta \in \mathbb{Z}} \widetilde{Kh}_\delta(L) \qquad \widehat{HFK}(L) = \bigoplus_{\delta \in \mathbb{Z}} \widehat{HFK}_\delta(L)$$

<u>definition:</u>

We call a graded vector space

$$V = \bigoplus_{\delta \in \mathbb{Z}} V_\delta$$

thin if $V_\delta = 0$ for all but one δ.

We will work over $\mathbb{F}_2 = \mathbb{Z}/2$.

baby theorem: (more precise)

Let $L = P_{2,-3} \cup_\varphi P_{2,-3}$. Then

$\widetilde{Kh}(L;\mathbb{F}_2)$ is thin $\Leftrightarrow$ $\widehat{HFK}(L;\mathbb{F}_2)$ is thin.

theorem: [Lee, Ozváth-Szabó]

If L is an alternating link, then

$\widetilde{Kh}(L)$ and $\widehat{HFK}(L)$ are thin.

theorem: [Dowlin]

$\exists$ δ-grading preserving spectral

sequence from $\widetilde{Kh}(L;\mathbb{Q})$ to $\widehat{HFK}(L;\mathbb{Q})$.

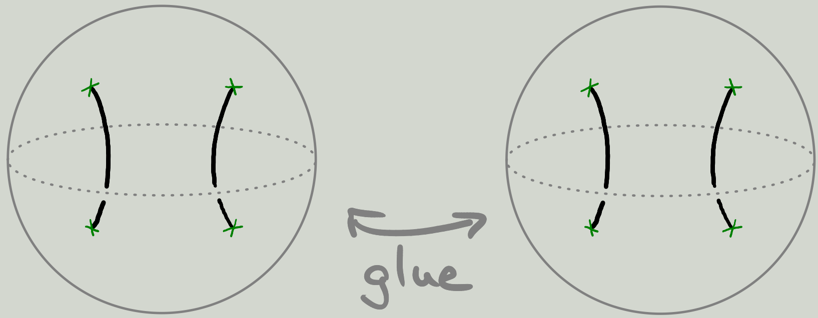
question:

What happens when we replace $P_{2,-3}$

by other tangles?

§ 3 Rational tangles

Let us replace $P_{2,-3}$ by a trivial tangle:



$$(S^2, 4 \text{ points}) \xrightarrow{\varphi} (S^2, 4 \text{ points})$$

We may consider φ up to homotopy, i.e.

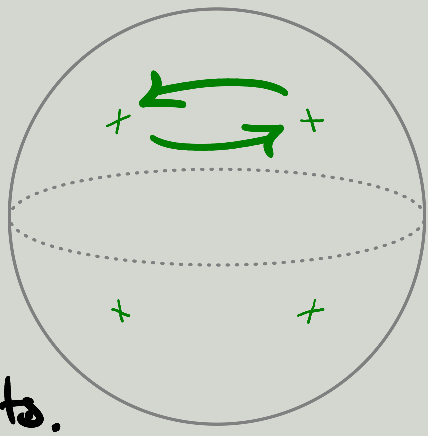
$$\varphi \in \mathrm{Mod}(S^2, 4 \text{ points})$$

mapping class group

<u>fact:</u>

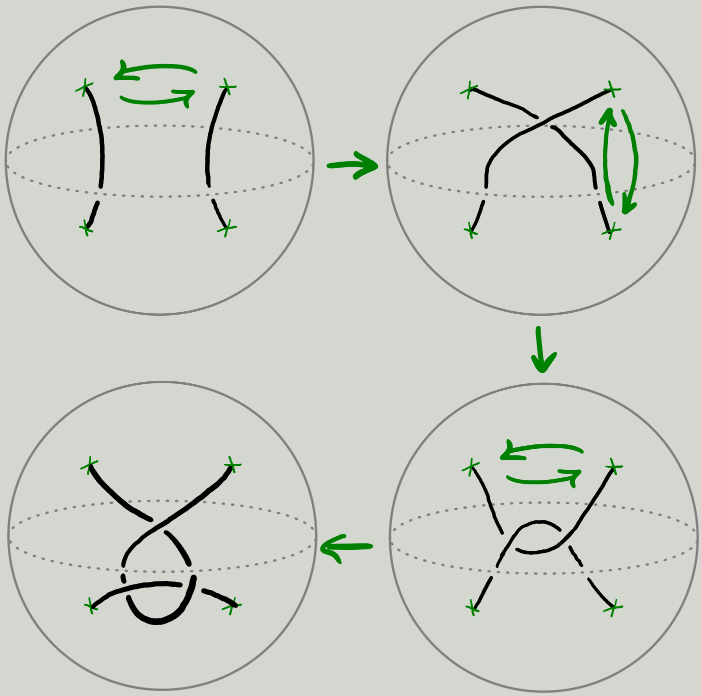
$\mathrm{Mod}(S^2, 4 \text{ points})$

is generated by twists.



So φ acts on Conway tangles by adding twists:

example:

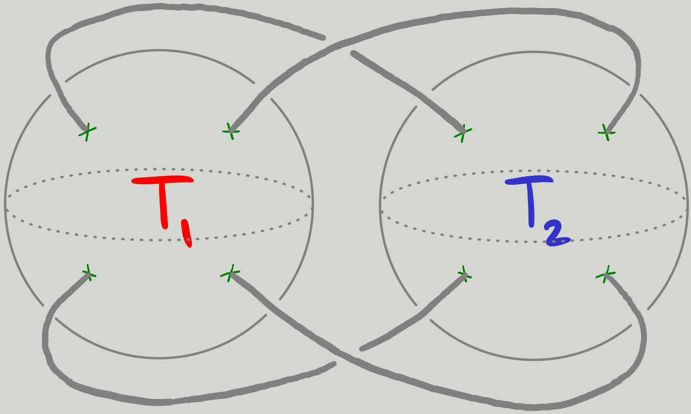


definition:

A tangle is called rational if it can be obtained from the trivial tangle by applying some $\varphi \in \mathrm{Mod}(S^2, 4 \text{ points})$.

definition:

The union of two Conway tangles:

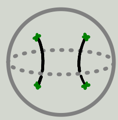
$T_1 \cup T_2 :=$ 

lemma:

$$T_1 \cup T_2 = T_2 \cup T_1 .$$

baby theorem for the trivial tangle:

Let $L = T_1 \cup T_2$ where

T_1 = trivial tangle 

T_2 = rational tangle.

Then

$$\widetilde{Kh}(L;\mathbb{F}_2) \text{ is thin} \Leftrightarrow \widehat{HFK}(L;\mathbb{F}_2) \text{ is thin}.$$

question:

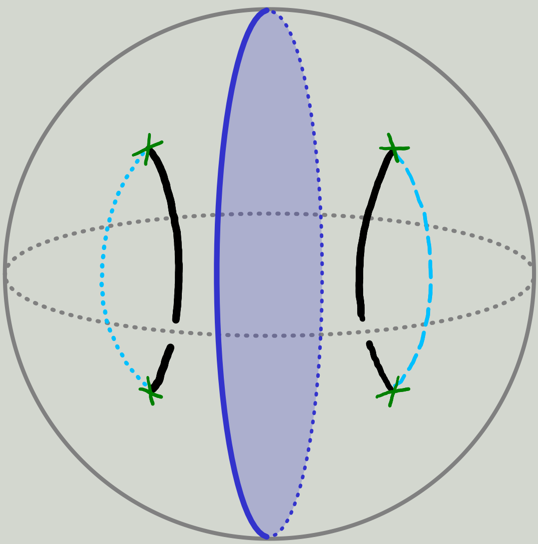
How many rational tangles are there?

lemma:

There is a 2:1 correspondence

$$\{\text{embedded arcs } (I,\partial I) \hookrightarrow (S^2, 4 \text{ points})\}$$
$$\downarrow$$
$$\{\text{rational tangles}\}$$

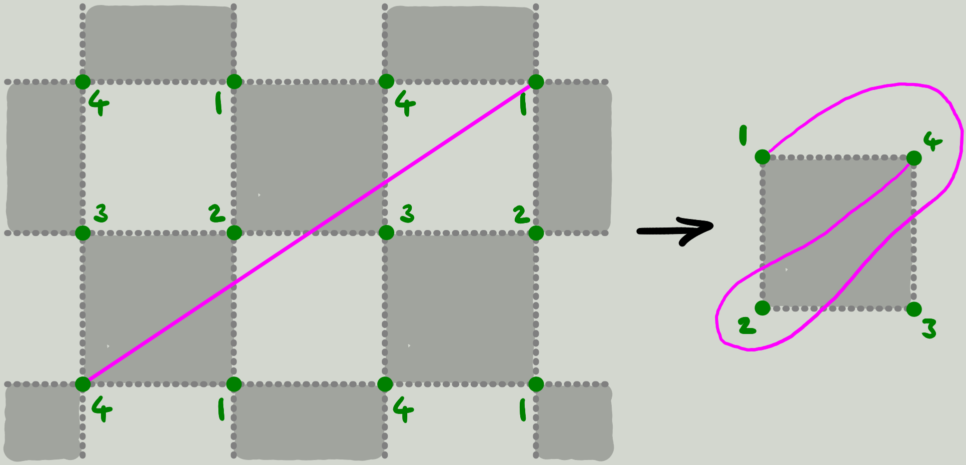
↳ **sketch proof:**



$\blacksquare$

Consider the covering

$$\mathbb{R}^2 \setminus \mathbb{Z}^2 \longrightarrow S^2 \setminus (4 \text{ points})$$

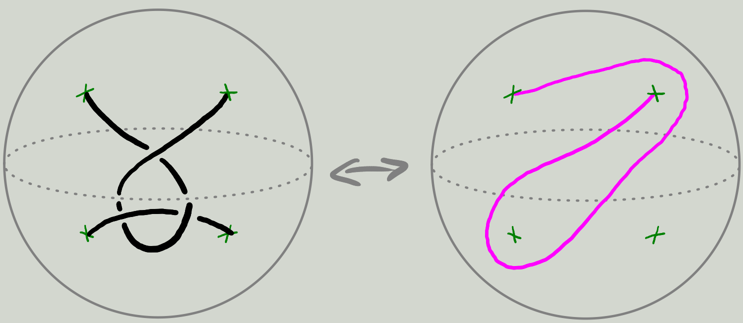


theorem: [Conway]

Rational tangles are classified by $\mathbb{Q}P^1$.

$\mathbb{Q} \cup \{\infty\}$

example:



definition:

Write $Q_{p/q}$ for the rational tangle corresponding to the slope $p/q \in \mathbb{QP}^1$.

baby theorem for the trivial tangle: (restated)

Let $L = Q_\infty \cup Q_{p/q}$ for some $p/q \in \mathbb{QP}^1$.

Then

$\widetilde{Kh}(L;\mathbb{F}_2)$ is thin $\Leftrightarrow$ $\widehat{HFK}(L;\mathbb{F}_2)$ is thin.

↳ proof:

$p/q \neq \infty$: L is alternating, so both $\widetilde{Kh}(L;\mathbb{F}_2)$ and $\widehat{HFK}(L;\mathbb{F}_2)$ are thin.

$p/q = \infty$: L = 2-component unlink, so neither $\widetilde{Kh}(L;\mathbb{F}_2)$ nor $\widehat{HFK}(L;\mathbb{F}_2)$ is thin. $\blacksquare$

§4 Thin Rational Fillings

definition:

Given a Conway tangle T and some slope $p/q \in \mathbb{Q}P^1$, let

$$T(p/q) := Q_{-p/q} \cup T$$

We then define

$$\Theta_{HF}(T) := \left\{ p/q \in \mathbb{Q}P^1 \mid \widehat{HFK}(T(p/q)) \text{ is thin} \right\}$$

$$\Theta_{Kh}(T) := \left\{ p/q \in \mathbb{Q}P^1 \mid \widetilde{Kh}(T(p/q)) \text{ is thin} \right\}$$

Reminder: We work over $\mathbb{F}_2 = \mathbb{Z}/2$.

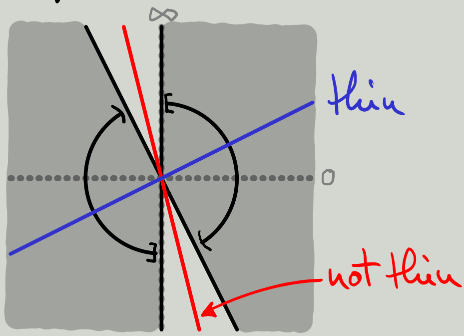
example:

$$\Theta_{HF}(Q_\infty) = \Theta_{Kh}(Q_\infty) = \mathbb{Q}P^1 \setminus \{\infty\}$$

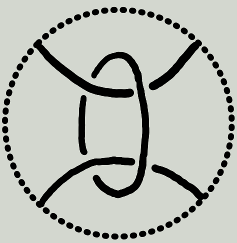
more examples: (proofs later)

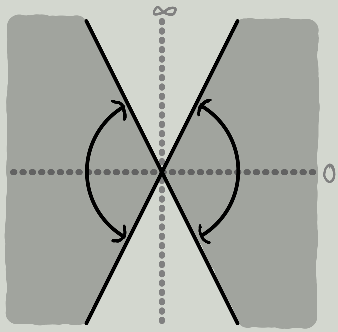
1) $\Theta_{HF}(P_{2,-3}) = \Theta_{Kh}(P_{2,-3}) = (-2,\infty]$

$P_{2,-3} =$ 



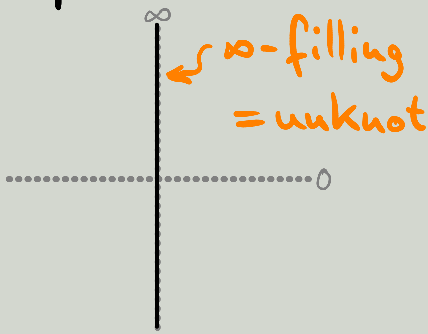
2) $\Theta_{HF}(P_{2,-2}) = \Theta_{Kh}(P_{2,-2}) = (-2,2)$

$P_{2,-2} =$ 



3) $\Theta_{HF}(T_{4,1}) = \Theta_{Kh}(T_{4,1}) = \{\infty\}$

$T_{4,1} =$ 



theorem A: [Kotelskiy-Watson-Z]

For any Conway tangle T, $\Theta_{HF}(T)$ is equal to one of the following:

a) $\emptyset$

b) a single point

c) two points (no known example)

d) an interval
   - open
   - half-open
   - closed

e) $\mathbb{Q}P^1 \setminus$ point

The same is true for $\Theta_{Kh}(T)$.

theorem B: [Kotelskiy-Watson-Z]

Let T_1 and T_2 be two Conway tangles. Suppose

interiors of Θ_{HF}

$$\left(-\mathring{\Theta}_{HF}(T_1)\right) \cup \mathring{\Theta}_{HF}(T_2) = \mathbb{Q}P^1.$$

Then $\widehat{HFK}(T_1 \cup T_2)$ is thin.

The same is true for $\widetilde{Kh}$ and Θ_{Kh}.

remark:

One can generalize theorem B to an exact criterion for thinness; see theorem 1.15 in our paper.

§ 5 The multicurve invariant HFT

$$\left\{\begin{matrix}\text{Conway tangles}\\ T \subset D^3\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{multicurves on}\\ \partial D^3 \smallsetminus \partial T\end{matrix}\right\}$$

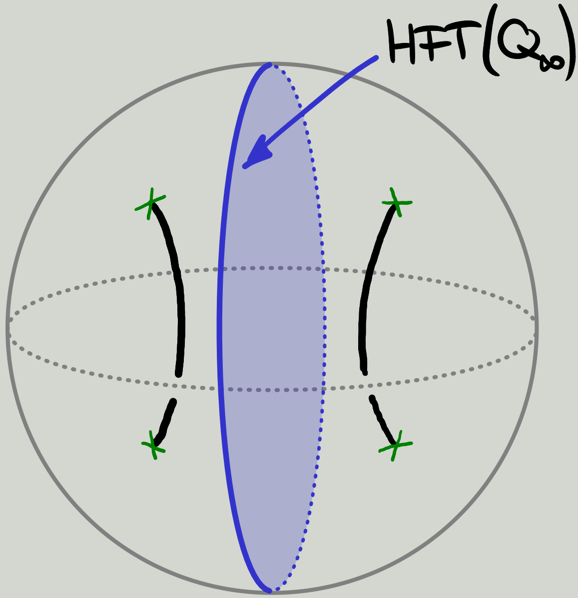
4-punctured sphere

$$T \longmapsto \operatorname{HFT}(T)$$

multicurve = finite set of immersed curves*

examples:

1) $T = Q_\infty$



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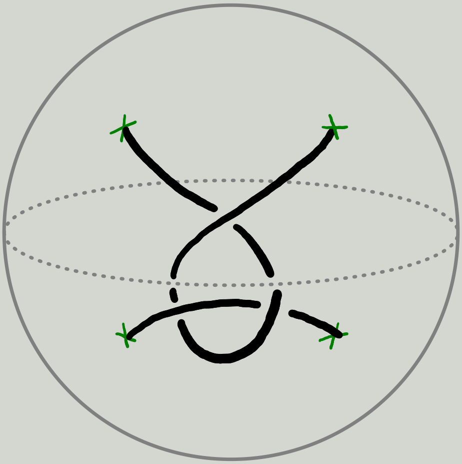
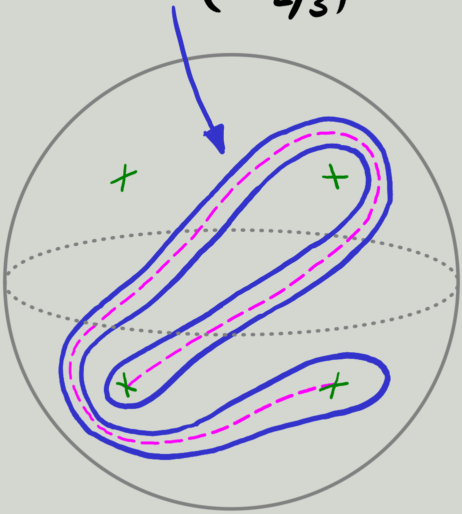
*plus local systems

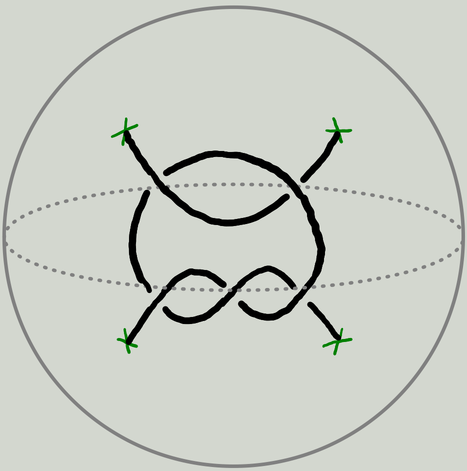
2) $T = Q_{2/3}$



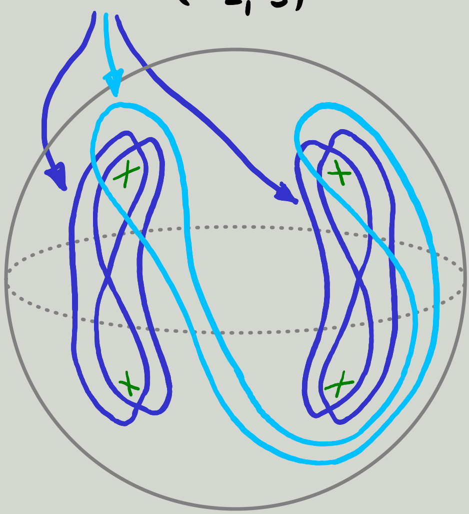
$HFT(Q_{2/3})$



3) $T = P_{2,-3}$



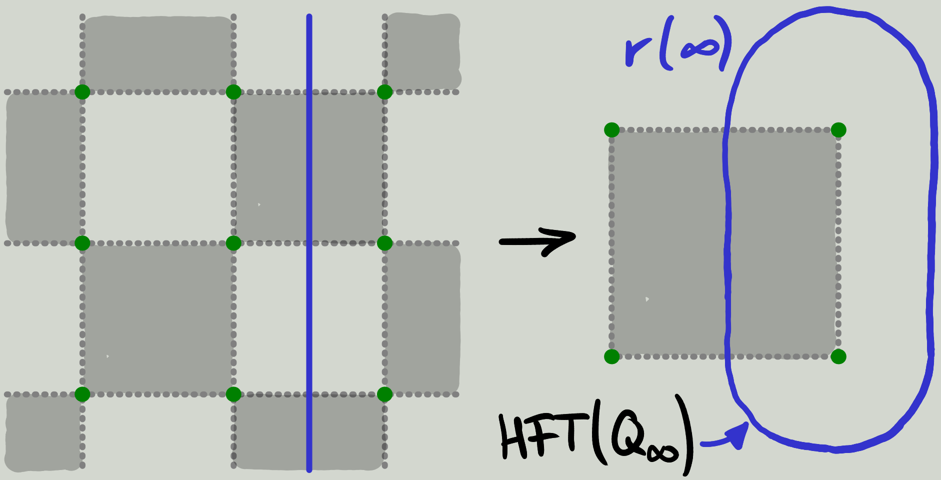
$HFT(P_{2,-3})$

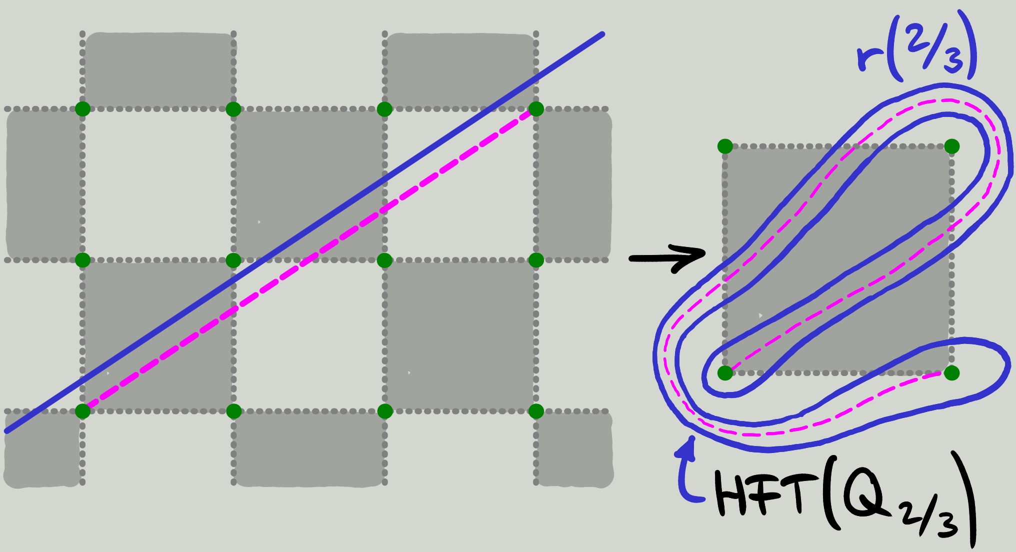


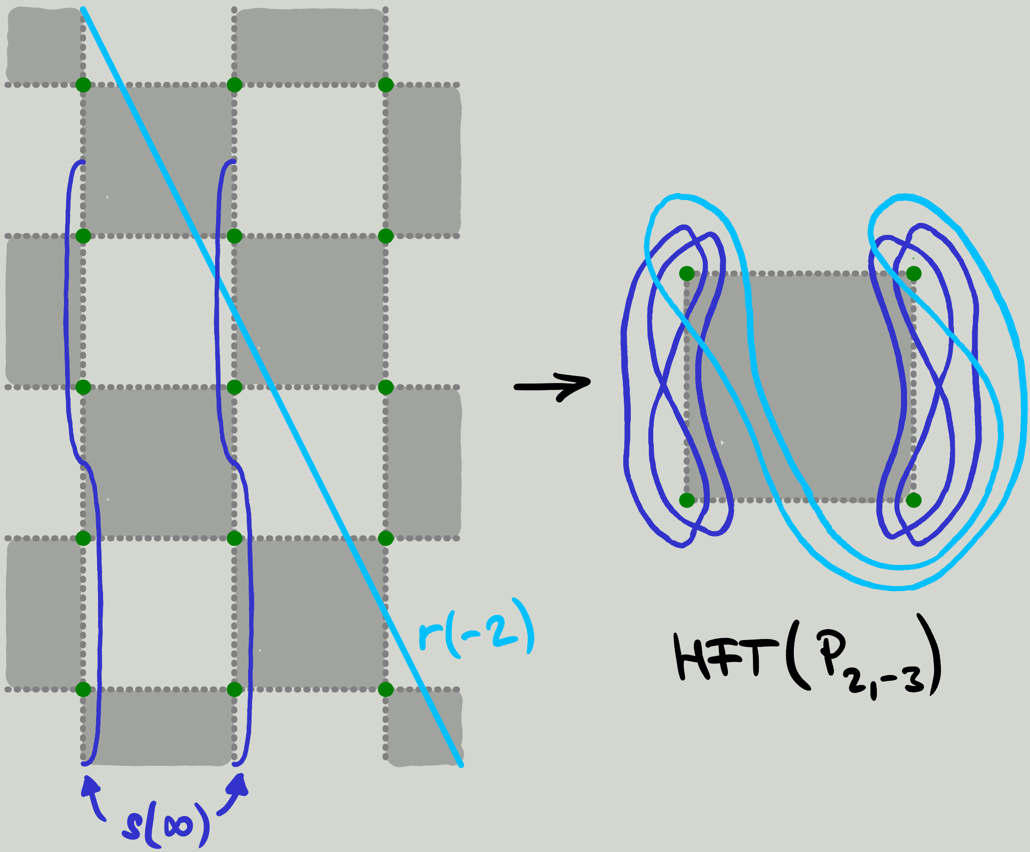
§ 6 Properties of HFT

Lift these curves along the covering

$$\mathbb{R}^2 \setminus \mathbb{Z}^2 \longrightarrow S^2 \setminus (4 \text{ points})$$







r(-2)
s(∞)
HFT(P_{2,-3})

theorem: (geography of HFT) [Z]

All components of HFT(T) are linear. In fact, for each slope $p/q \in \mathbb{QP}^1$, there are only two types of curves,*

namely

a) rational curves $r(p/q)$, and

b) special curves $s(p/q)$.

these consist of two components, like $s(\infty)$ in $HFT(P_{2,-3})$

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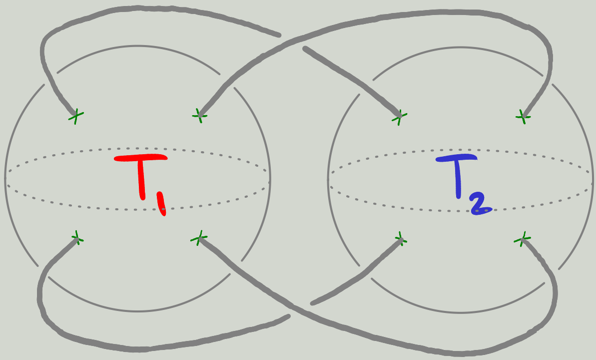
*up to local systems for rationals and length for specials

theorem: (gluing) [Z]

Suppose

$K := T_1 \cup T_2 =$ 

is a knot and

$\gamma_1 = -HFT(T_1)$ and

$\gamma_2 = HFT(T_2)$.

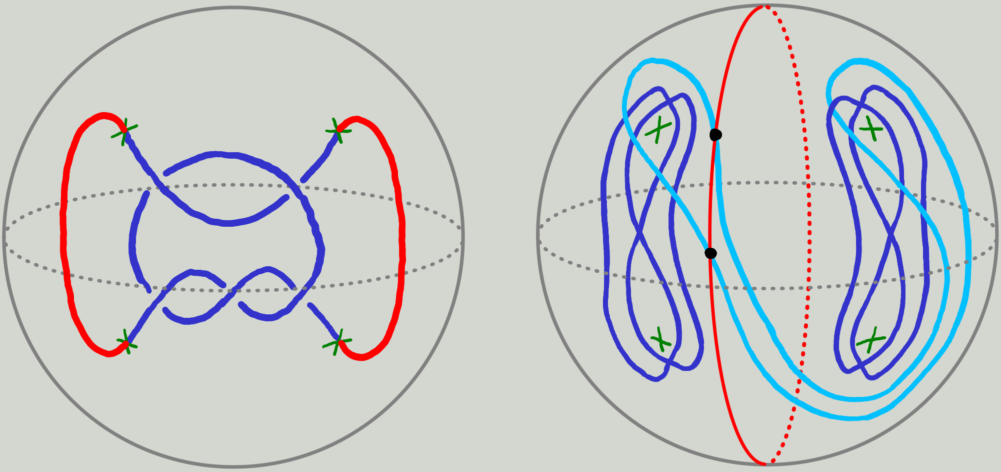
Then

$$\widehat{HFK}(K) \otimes \mathbb{F}_2^2 \cong HF(\gamma_1, \gamma_2)$$

Lagrangian Floer homology $\cong \mathbb{F}_2^d$

where $d = \min \#(\gamma_1 \cap \gamma_2)$

example:



§ 7 The δ-grading on HFT

HFT can be equipped with a bigrading.

lemma: [Kotelskiy-Watson-Z]

Let γ, γ' be two linear curves of slopes $\sigma(\gamma) \neq \sigma(\gamma')$.
Then $HF(\gamma, \gamma')$ is thin.

definition:

If $\sigma(\gamma) \neq \sigma(\gamma')$, define

$\delta(\gamma,\gamma') := \delta$-grading of $HF(\gamma,\gamma')$ $\neq 0$

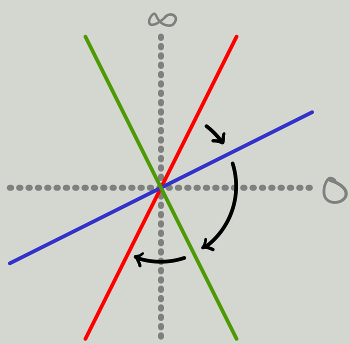
lemma: (anti-symmetry) [KWZ]

$$\delta(\gamma,\gamma') = 1 - \delta(\gamma',\gamma)$$

lemma: (transitivity) [KWZ]

$$\delta(\gamma,\gamma'') = \delta(\gamma,\gamma') + \delta(\gamma',\gamma'')$$

if $\sigma(\gamma) > \sigma(\gamma') > \sigma(\gamma'') > \sigma(\gamma)$.



example:

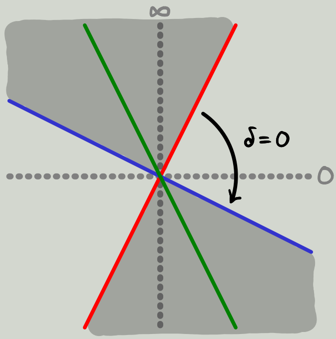
Let $\Gamma = \{\gamma, \gamma'\}$ with $\sigma(\gamma) \neq \sigma(\gamma')$ and γ'' such that $\sigma(\gamma) \neq \sigma(\gamma'') \neq \sigma(\gamma')$.

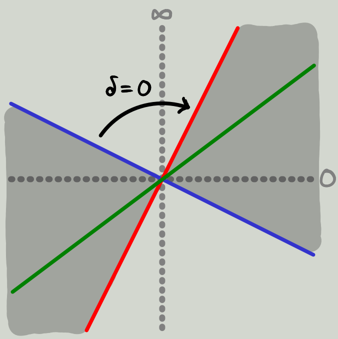
Then

$$HF(\Gamma, \gamma'') \text{ is thin}$$

$$\Leftrightarrow \delta(\gamma, \gamma'') = \delta(\gamma', \gamma'')$$

$$\Leftrightarrow \begin{cases} \delta(\gamma, \gamma') = 0 \text{ if} \\ \delta(\gamma', \gamma) = 0 \text{ if} \end{cases}$$





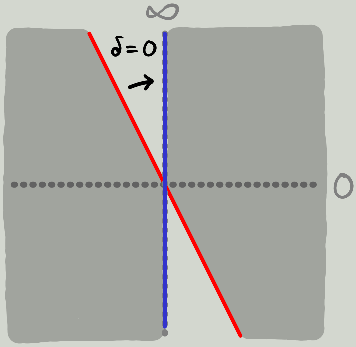
$\Updownarrow$ anti-symmetry

$$\delta(\gamma, \gamma') = 1$$

example:

$$\Gamma = \mathrm{HFT}(P_{2,-3}) = \{r(-2), s(\infty)\}$$

where $\delta(r(-2), s(\infty)) = 0$



Therefore

$$\Theta_{HF}(P_{2,-3}) = \left\{ p/q \in \mathbb{Q}P^1 \;\middle|\; \begin{array}{c} \widehat{\mathrm{HFK}}(P_{2,-3}(p/q)) \\ \text{is thin} \end{array} \right\}$$

by the Gluing Theorem

$$= \left\{ p/q \in \mathbb{Q}P^1 \;\middle|\; \begin{array}{c} \mathrm{HF}(r(p/q), \Gamma) \\ \text{is thin} \end{array} \right\}$$

is an interval $\langle -2, \infty \rangle$.

lemma: (endpoint behaviour) [KWZ]

For any slope $p/q \in \mathbb{Q}P^1$,

a) $HF(r(p/q), r(p/q))$ is not thin

b) $HF(s(p/q), s(p/q))$ is not thin

c) $HF(r(p/q), s(p/q)) = 0$

d) $HF(s(p/q), r(p/q)) = 0$

example:

$$\Theta_{HF}(P_{2,-3}) = (-2, \infty]$$

§ 8 The multicurve invariant $\widetilde{Kh}$

$$\left\{\begin{matrix}\text{Conway tangles}\\ T\subset D^3\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{multicurves on}\\ S^2\smallsetminus(4\text{ points})\end{matrix}\right\}$$

$$T \longmapsto \widetilde{Kh}(T)$$

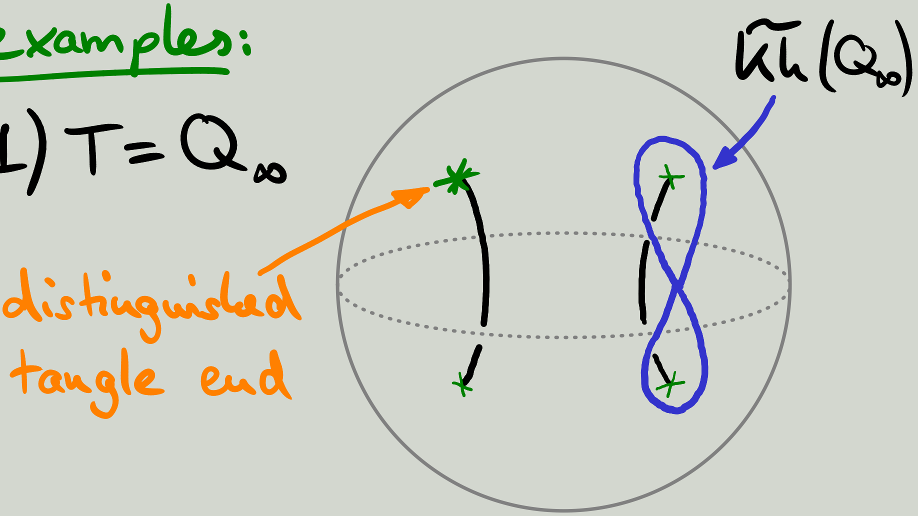
remark:

If we choose a distinguished tangle end, there is a natural identification

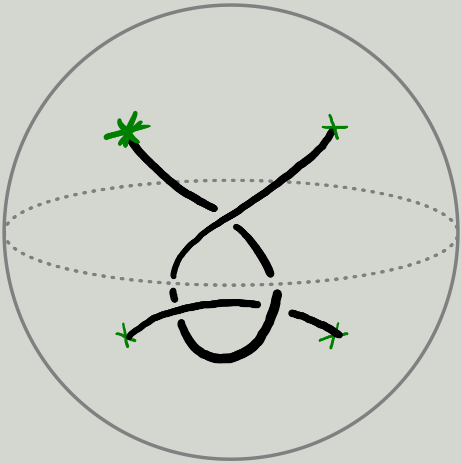
$$S^2\smallsetminus(4\text{ points}) \cong \partial D^3\smallsetminus\partial T$$

examples:

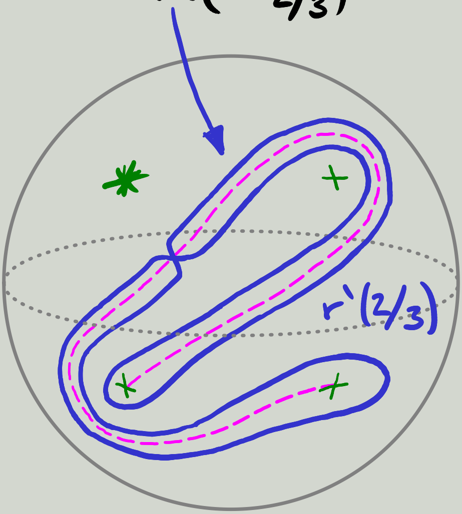
1) $T=Q_\infty$



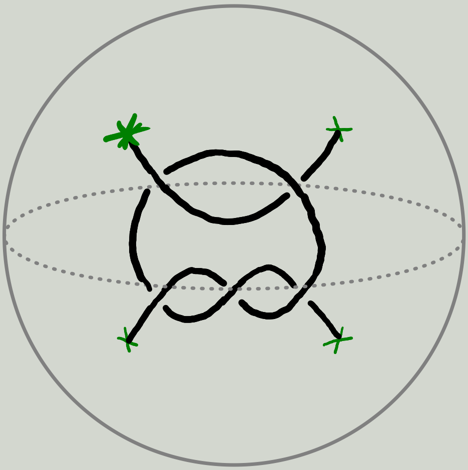
2) $T = Q_{2/3}$



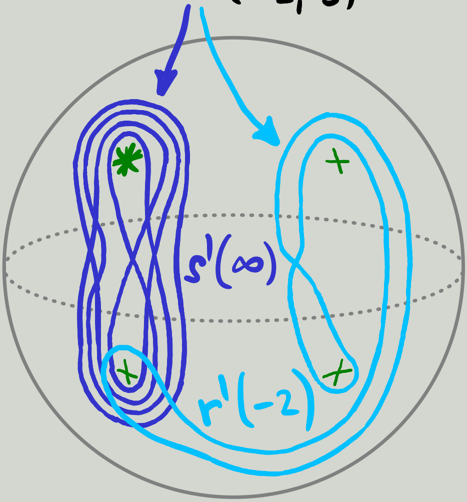
$\widetilde{Kh}(Q_{2/3})$



3) $T = P_{2,-3}$



$\widetilde{Kh}(P_{2,-3})$



Theorem: (gluing) [Kotelskiy-Watson-Z]

$$\widetilde{Kh}(T_1 \cup T_2) \otimes \mathbb{F}_2^2 \cong HF\left(-\widetilde{Kh}(T_1), \widetilde{Kh}(T_2)\right)$$

theorem: (geography of $\widehat{\widetilde{Kh}}$) [KWZ]

Every component of $\widehat{\widetilde{Kh}}(T)$ belongs to one of two families of curves*, namely

a) rational curves $\tilde{r}(p/q)$, and

b) special curves $\tilde{s}(p/q)$,

where $p/q \in \mathbb{Q}P^1$.

example:

$$\widehat{\widetilde{Kh}}(P_{2,-3}) = \{\tilde{r}(-2), \tilde{s}(\infty)\}$$

Compare with

$$\mathrm{HFT}(P_{2,-3}) = \{r(-2), s(\infty)\}$$

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*up to length

theorem: [Kotelskiy-Watson-Z]

The δ-grading on $\widetilde{Kh}$ has the same formal properties as the δ-grading on HFT. In particular:

a) $HF(\gamma, \gamma')$ is thin if $\sigma(\gamma) \neq \sigma(\gamma')$.

b) δ is anti-symmetric.

c) δ is transitive.

d) δ has the same endpoint behaviour.

baby theorem: [Kotelskiy-Watson-Z]

Let $L = P_{2,-3} \cup_\varphi P_{2,-3}$. Then $\widetilde{Kh}(L)$ is thin $\Leftrightarrow$ $\widehat{HFK}(L)$ is thin.

$\blacksquare$