Resurgence and quantum topology

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based mostly on work with Stavros Garoufalidis, Jie Gu and Campbell Wheeler

Motivation

In this talk I will consider a particular type of topological invariants for knots, given by power series with factorial growth:

$$\varphi(z) = \sum_{n\geq 0} a_n z^n \qquad a_n \sim n!$$

This series are also called Gevrey-1 series, and they have zero radius of convergence. They appear often in quantum mechanics and in quantum field theory, in the context of perturbation theory.

Making sense of these series is a delicate task. A general and ambitious framework for doing this was constructed in the 1980s by the French mathematician Jean Ecalle. It is called the **theory of resurgence**

It builds on the contributions of many physicists, in particular Carl Bender, T.T. Wu, André Voros, Giorgio Parisi and Jean Zinn-Justin, who were studying these series in quantum theory

In quantum topology, one often uses physics ideas and techniques to obtain invariants of knots and three-manifolds. In this talk I will use the theory of resurgence to shed light on invariants of knots that appear naturally in quantum topology and are given by Gevrey-1 series (this is a program sketched by Stavros Garoufalidis long ago).

We will see that, thanks to the theory of resurgence, one finds "hidden" structures inside these perturbative series, and in particular **integer invariants of knots** (cf. Sergei Gukov's talk).

Perturbative series for knots

Let K be a hyperbolic knot in the 3-sphere. It is possible to define a finite collection of Gevrey-1 formal power series

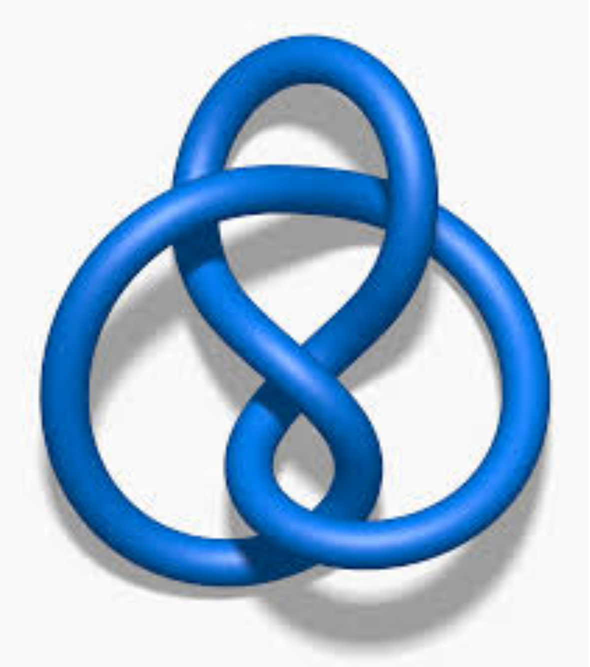
$$\varphi_\sigma(\tau) = \sum_{n \geq 0} a_n^\sigma \tau^n$$

which are topological invariants of K

What is σ? Let ρ be a flat SL(2,C) connection on the complement of K. It can be labelled by a complex variable u (corresponding to a holonomy around the knot) and a discrete variable, which is my σ.

Example: 4_1

the figure eight knot



$$\sigma = 0, g, c$$

respectively: the **trivial** connection, the **geometric** connection (corresponding to the complete Riemannian metric of constant negative curvature), and its **conjugate**

In this talk, I'm setting *u=0*. Everything I will say can be generalized to arbitrary *u*

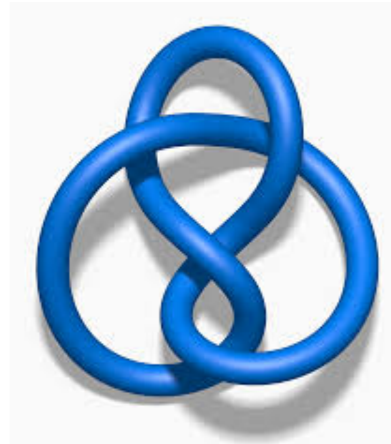
Let me sketch how to define the series $\varphi_0(\tau)$

Let us consider the **colored Jones polynomial** of *K*

$$J_n(q) \qquad n \in \mathbb{Z}_{>0}$$

It can be computed combinatorially for any knot and positive integer *n*, and it satisfies:

$$J_1(q) = 1 \qquad J_2(q) = J(q)$$

Example: 

$$J_n(q) = \sum_{m=0}^{n-1} (-1)^m q^{-m(m+1)/2} \prod_{j=1}^{m} (1 - q^{j-n})(1 - q^{j+n})$$

In order to obtain a series, we need a small parameter. We then set

$$q = \mathrm{e}^{2\pi \mathrm{i} \tau}$$

and expand in τ

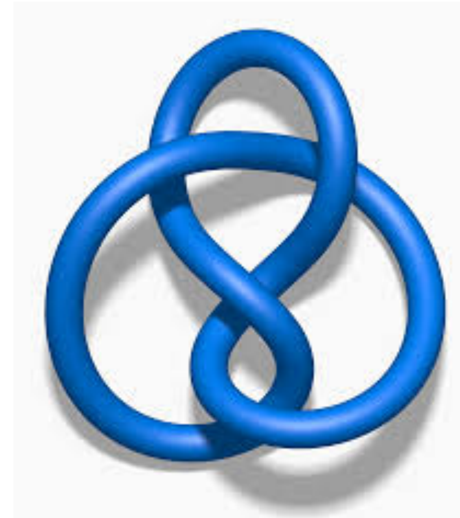
For any positive integer *n*, we obtain a convergent series

$$J_n(q) \sim \sum_{m \geq 0} c_m(n) \tau^m$$

The coefficients $c_m(n)$ turn out to be polynomials in *n*, and, somewhat surprisingly, when we evaluate them at *n=0* we obtain a Gevrey-1 series much studied by Melvin, Morton, Rozansky, and others

$$\varphi_0(\tau) = \sum_{m \geq 0} c_m(0) \tau^m$$

Example:



$$\varphi_0(\tau) = 1 + 4\pi^2\tau^2 + \frac{188\pi^4\tau^4}{3} + \frac{98888\pi^6\tau^6}{45} + \cdots$$

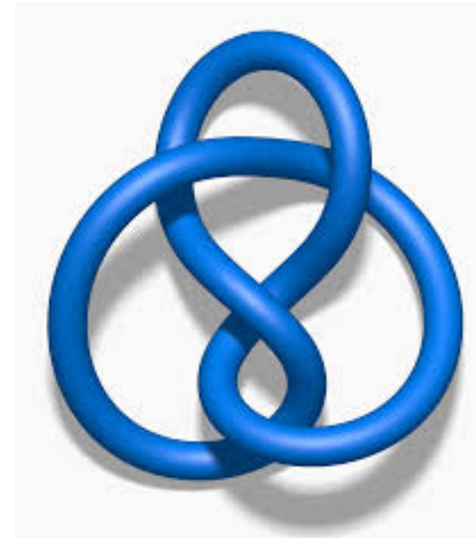
Rozansky gave a physical interpretation of this series, as an expansion around the trivial connection of the path integral of SU(2) CS-Witten theory on the complement of *K*. So it is a typical, factorially divergent QFT series

For the other flat connections, the Gevrey-1 series can be defined in various ways. From a physics point of view, their construction was explained in work by [Dimofte-Gukov-Lenell-Zagier]

Mathematically, they can be defined as formal asymptotic expansions of the **Andersen-Kashaev invariant** of the complement of *K*, around non-trivial flat connections. There is also a direct construction by [Dimofte-Garoufalidis] which gives (conjecturally) the same answer.

They should be regarded as expansions of the path integral of SL(2,C) CS-Witten theory around non-trivial flat connections.

Example: 4_1



$$\varphi_g\left(\frac{\tau}{2\pi\mathrm{i}}\right)=\frac{1}{3^{1/4}}\left(1+\frac{11\tau}{72\sqrt{-3}}+\frac{697\tau^2}{2\,(72\sqrt{-3})^2}+\frac{724351\tau^3}{30\,(72\sqrt{-3})^3}+\cdots\right)$$

=all-orders asymptotic expansion of the Kashaev invariant

$$\varphi_c(\tau)=\varphi_g(-\tau)$$

One could think that all these formal power series are independent. For example $\varphi_0(\tau)$ and the series labelled from non-trivial flat connections come from very different invariants. It turns out that this is not the case: surprisingly, they know about each other!

This is actually the meaning of "resurgence": one series "resurges" in the others. To understand this precisely, we need some "resurgent technology"

Decoding divergent series

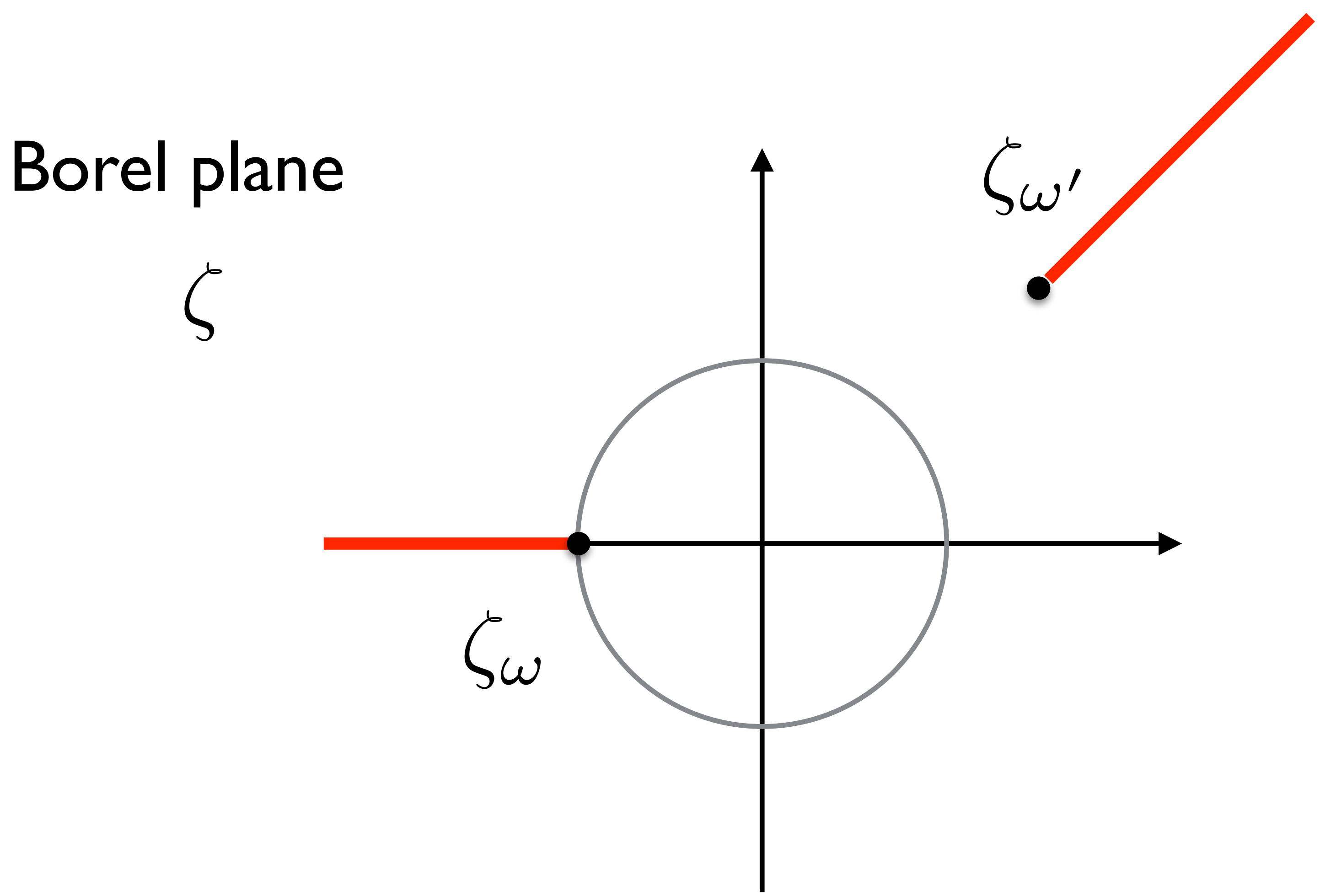
Let us consider a formal power series with factorially growing coefficients

$$\varphi(z) = \sum_{n\geq 0} a_n z^n \qquad a_n \sim n!$$

The first step in resurgent analysis is the **Borel transform**, a deceptively simple way of obtaining "nice" functions

$$\boxed{\varphi(z) = \sum_{n\geq 0} a_n z^n} \xrightarrow{\boxed{\text{Borel transform}}} \boxed{\widehat{\varphi}(\zeta) = \sum_{n\geq 0} \frac{a_n}{n!} \zeta^n}$$

The Borel transform $\widehat{\varphi}(\zeta)$ is by definition analytic at the origin. Very often it can be analytically continued to the complex plane, displaying a set of **singularities** (poles, branch cuts)



As we will see, these singularities contain the additional information "hidden" in the original Gevrey-1 series

To extract this information, we have to consider the expansion of the Borel transform around each singularity. These leads to **new formal power series**.

Let us consider for simplicity the so-called **simple resurgent functions**, where singularities are logarithmic branch cuts. The expansion around a singularity at $\zeta = \zeta_\omega$ has the form

$$\widehat{\varphi}(\zeta) = -\mathsf{S}_\omega \, \widehat{\varphi}_\omega(\zeta - \zeta_\omega) \frac{\log(\zeta - \zeta_\omega)}{2\pi \mathrm{i}} + \text{regular}$$

The function $\widehat{\varphi}_\omega(\xi)$ is typically analytic at the origin

$$\widehat{\varphi}_\omega(\xi) = \sum_{n\geq 0} \widehat{a}_{n,\omega}\xi^n$$

but we can think about it as the Borel transform of a **new Gevrey-1 series** associated to the singularity:

$$\varphi_\omega(z) = \sum_{n\geq 0} a_{n,\omega} z^n \qquad a_{n,\omega} = n!\,\widehat{a}_{n,\omega}$$

The constant S_ω is called a **Stokes constant** and plays an important role in the theory. Its value depends on the normalization of $\varphi_\omega(z)$

We can repeat the same analysis for the new power series found in this way, and generate further series. At the end, we obtain **a set of formal power series** associated to the original power series, which I call the "minimal resurgent structure" associated to $\varphi(z)$

$$\varphi(z) \longrightarrow \mathfrak{B}_{\varphi} = \{\varphi_{\omega}(z)\}_{\omega \in \Omega}$$

We also have a **matrix of Stokes constants** defined by

$$\widehat{\varphi}_{\omega}(\zeta_{\omega'} + \xi) = -\mathsf{S}_{\omega\omega'} \frac{\log(\xi)}{2\pi\mathrm{i}} \widehat{\varphi}_{\omega'}(\xi) + \text{regular}$$

Therefore, starting with a single Gevrey-1 series, we find potentially a very rich structure!

An elementary example: the Airy functions

The formal power series underlying the Airy "Ai" function is

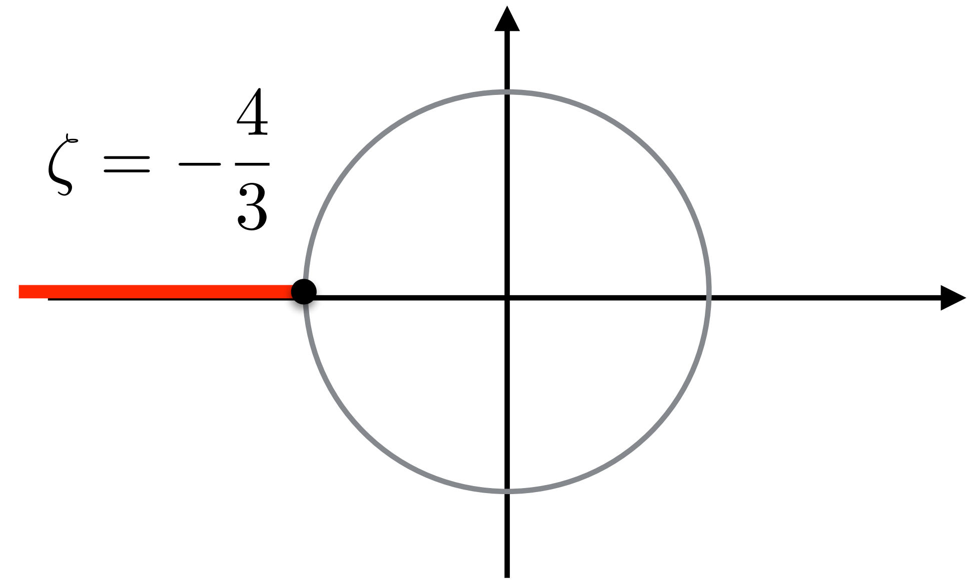
$$\varphi_1(z) = \sum_{n\geq 0} \frac{1}{2\pi}\left(-\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{n!} z^n$$

In this case the resurgent structure can be worked out in detail, since the Borel transform is simply

$$\widehat{\varphi}_1(\zeta) = {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1; -\frac{3\zeta}{4}\right)$$

This a simple resurgent function, with a log singularity along the negative real axis:

Borel plane:



By studying the expansion around the singularity, one finds the other formal power series involved in the theory, underlying the "Bi" Airy function

$$\varphi_2(z) = \sum_{n \geq 0} \frac{1}{2\pi} \left(\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{n!} z^n$$

The Stokes constants are $\mathsf{S}_{12} = \mathsf{S}_{21} = \mathrm{i}$

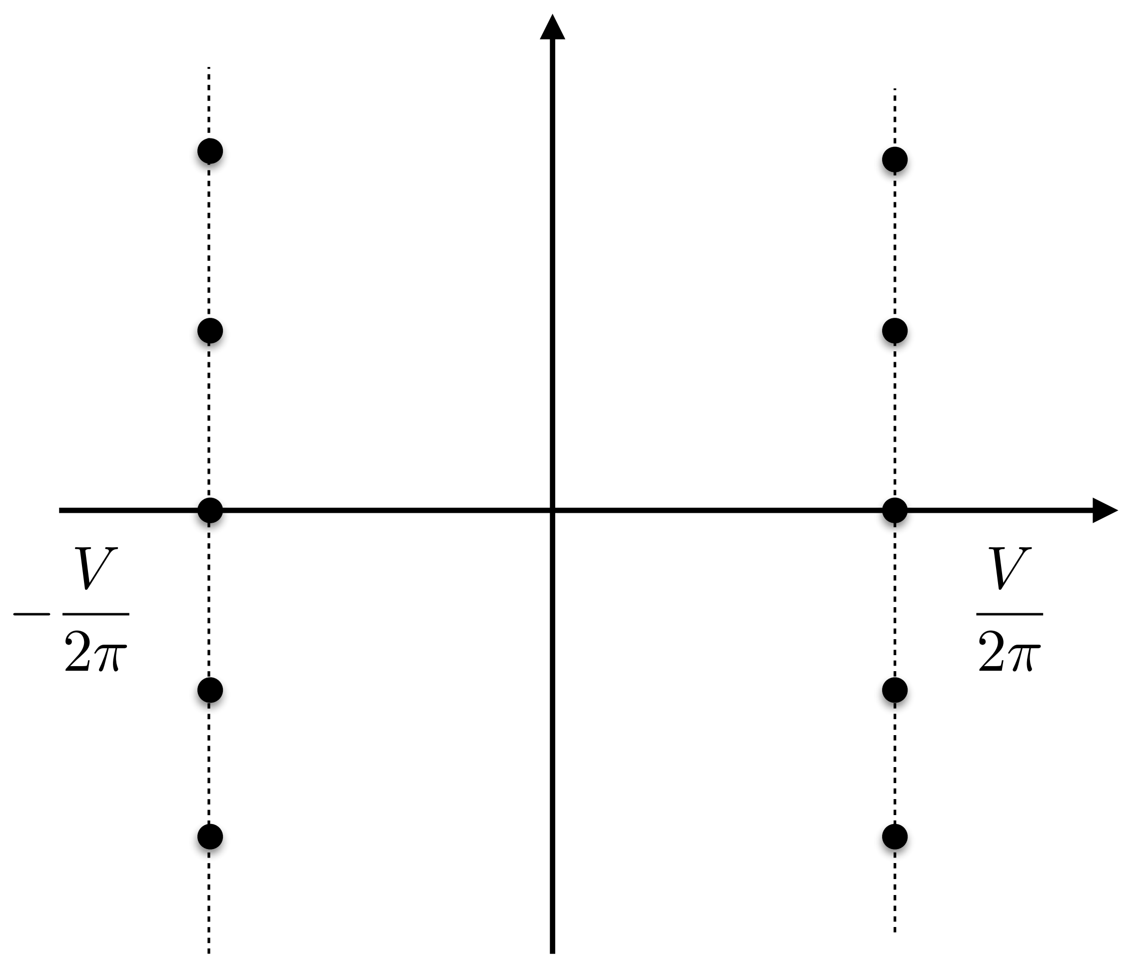
Resurgence of perturbative CS

We can now ask: what is the resurgent structure of the Gevrey-1 series associated to a hyperbolic knot K? Let me state some (mostly conjectural results) for the figure eight knot

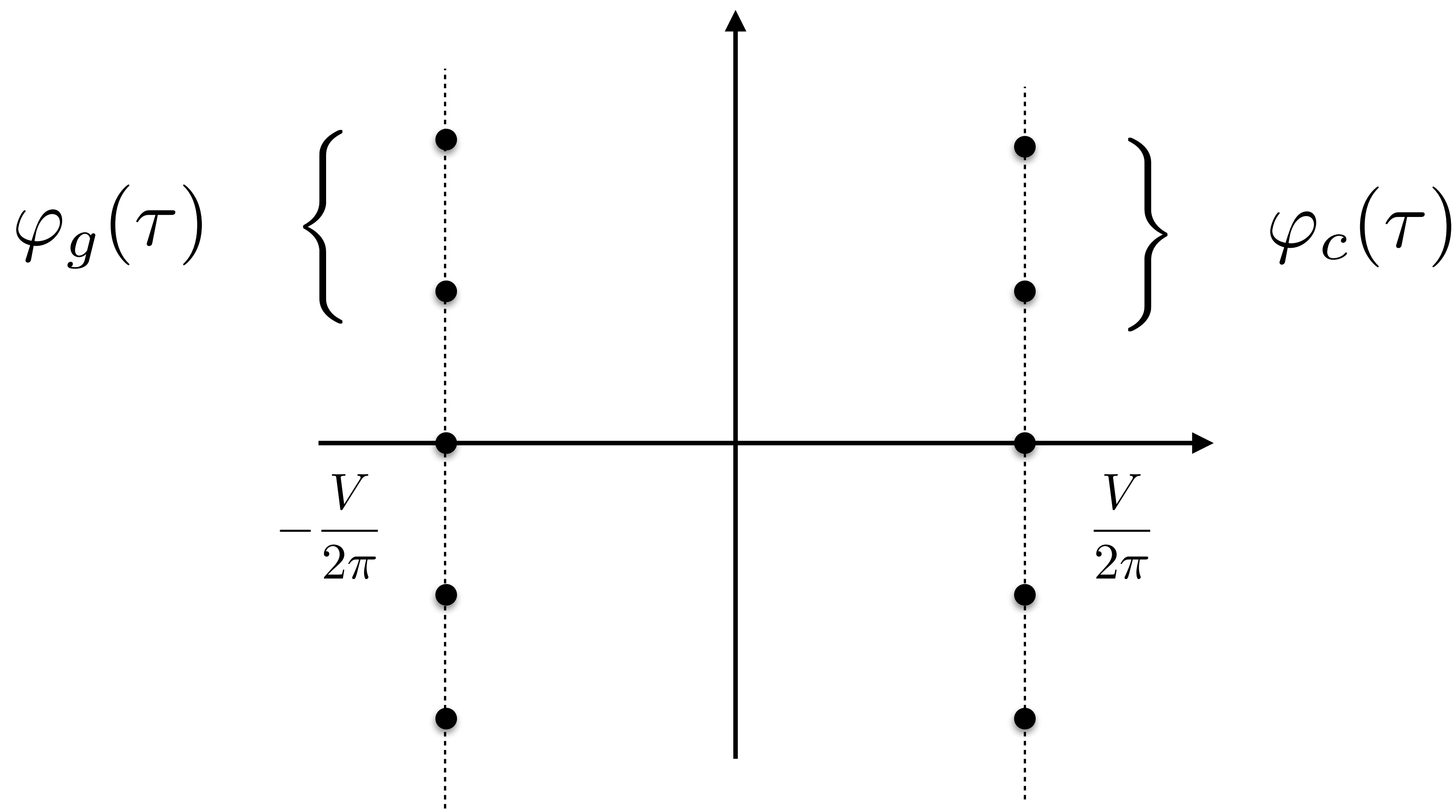
Conjecture [Garoufalidis-Gu-M.M.-Wheeler]: for the figure eight knot, the singularities in the Borel plane of $\varphi_0(\tau)$ are located at

$$\pm\frac{V}{2\pi}+2k\pi\mathrm{i} \qquad\qquad k\in\mathbb{Z}$$

V=2.02988… : hyperbolic volume of the complement of the knot.



It turns out that this is a simple resurgent function. We can then ask what are the formal power series associated to the singularities



So the Gevrey-1 series for the trivial connection “sees” the series labelled by non-trivial flat connections.

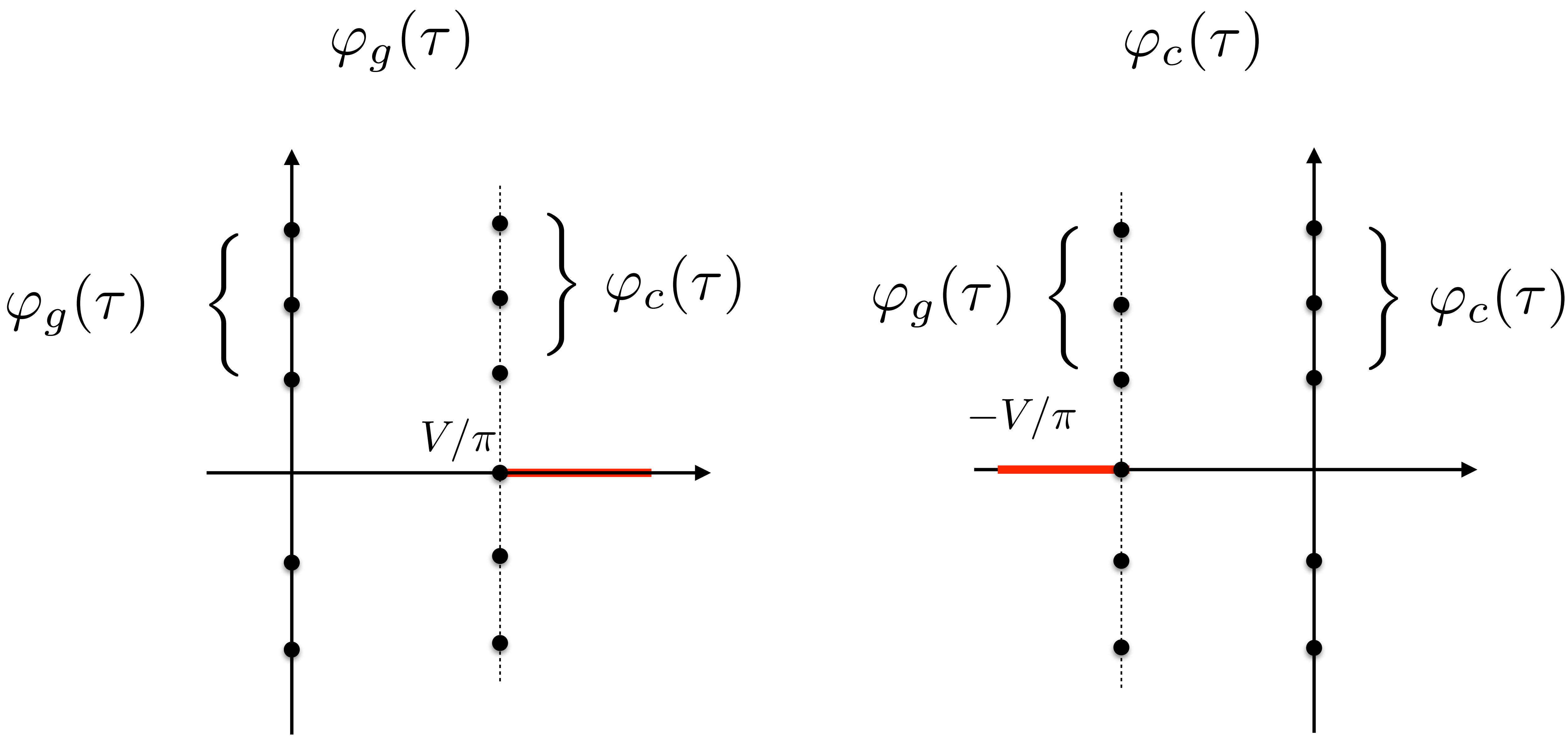
This can be also rephrased by saying that, through the resurgent structure of $\varphi_0(\tau)$, the colored Jones polynomial of a knot knows about its hyperbolic volume!

More precisely, our results suggest a **resurgent version of Kashaev's volume conjecture:**

For a hyperbolic knot, the Borel transform of the series $\varphi_0(\tau)$ has singularities at

$$\mathrm{Re}(\zeta) = \pm \frac{V}{2\pi}$$

We can now continue our resurgent analysis and look at the singularities in the Borel transforms of $\varphi_{g,c}(\tau)$, which are also simple resurgent functions



Although the tower of singularities lead to the same series, the corresponding Stokes constants are all **different** and they turn out to be **integer numbers**!

It is possible to gather these integers into a 3x3 matrix of formal *q*-series for each half-plane. One finds, for the upper half plane,

$$\begin{pmatrix} 1 & q-6q^2-22q^3-13q^4+O\left(q^5\right) & q+11q^2+24q^3+5q^4+O\left(q^5\right) \\ 0 & 1-8q-9q^2+18q^3+46q^4+O\left(q^5\right) & 9q+3q^2-39q^3-69q^4+O\left(q^5\right) \\ 0 & -9q-3q^2+39q^3+69q^4+O\left(q^5\right) & 1+8q-8q^2-64q^3-81q^4+O\left(q^5\right) \end{pmatrix}$$

This matrix represents a Stokes automorphism acting on

$$(\varphi_0(\tau), \varphi_g(\tau), \varphi_c(\tau))$$

It turns out that these q-series are closely related to invariants of hyperbolic knots which have been introduce recently.

The q-series involving the geometric/conjugate connection are closely related to the **Dimofte-Gaiotto-Gukov (DGG) index**, and the q-series in the first row turn out to be closely related to the **Gukov-Manolescu** invariant of the knot!

These (conjectural) connections between the Stokes constants and these integer invariants are highly non-trivial, mathematically. For the DGG index, a justification via categorification has been put forward recently by G. Moore and collaborators.

Conclusions

The theory of resurgence is a general framework to study factorially divergent series, with many applications in physics and mathematics. It unveils a rich structure in these series.

This theory, which is analytic in flavour, has recently been applied in geometry and topology, sometimes with surprising results. It sheds new light on quantum invariants of hyperbolic knots and on the volume conjecture.

This geometric version of resurgence theory has applications in other problems, like topological string theory. Many of these applications are still in its infancy and I expect exciting developments in this interdisciplinary field!

Thank you for your attention!